

PROJECTIVE DEFORMATIONS OF WEAKLY ORDERABLE HYPERBOLIC COXETER ORBIFOLDS

SUHYOUNG CHOI AND GYE-SEON LEE

ABSTRACT. A Coxeter n -orbifold is an n -dimensional orbifold which has the combinatorial type of a polyhedron. Each pair of adjacent $(n-1)$ -faces meet on an $(n-2)$ -face of some order m which is locally modeled on \mathbb{R}^n modulo the dihedral group of order $2m$ generated by two reflections. For $n \geq 3$, we study the deformation space of real projective structures on a Coxeter n -orbifold Q admitting a hyperbolic structure. Let f be the number of $(n-1)$ -faces, and let e_+ be the number of $(n-2)$ -faces of order ≥ 3 . A neighborhood of the hyperbolic structure in the deformation space is a smooth manifold of dimension $e_+ - n$ if the number of $(n-2)$ -faces of Q is equal to $nf - \frac{n(n+1)}{2}$, and Q is weakly orderable, ie the $(n-1)$ -faces of Q can be ordered so that each $(n-1)$ -face contains at most n $(n-2)$ -faces of order 2 in $(n-1)$ -faces of higher indices.

1. INTRODUCTION

In this paper, an *orbifold* is a quotient space Q of a manifold by a properly discontinuous discrete group action with an *orbifold structure*: Each point of Q has a neighborhood $\phi(U)$ that is a quotient of an open subset U of \mathbb{R}^n by a finite group action G with a chart $\phi: U \rightarrow \phi(U)$ inducing the quotient map $U \rightarrow U/G$. Two charts (U, G, ϕ) and (V, H, ψ) are *compatible* if each point of $\phi(U) \cap \psi(V)$ has an open neighborhood W' modeled on (W, K, ζ) with maps $W \rightarrow U$ and $W \rightarrow V$ lifting inclusions equivariant with respect to homomorphisms $K \rightarrow G$ and $K \rightarrow H$. An *atlas* is the collection of compatible charts of form (U, G, ϕ) so that the open sets of form $\phi(U)$ cover the space Q . An *orbifold structure* is a maximal atlas. A *singular point* is one where the finite group G is not trivial for any choice of such a chart (U, G, ϕ) , and the *singular locus* is the set of singular points.

A *Coxeter n -orbifold* is an n -dimensional orbifold whose underlying space is an n -dimensional domain and whose singular locus is in the boundary: Let \mathbb{A}^n be an n -dimensional affine space and let $P \subset \mathbb{A}^n$ be an n -dimensional *convex polytope*, ie the convex hull of a finite subset. The faces of codimension one and two are called the *facets* and *ridges*, respectively. Let P' be the subset of P by removing some of the faces whose codimension is greater than two. Denote by D_m the dihedral group of order $2m$. A Coxeter n -orbifold \hat{P} associated with P' is given as follows: the interior of each facet of P is silvered, ie the singular locus is locally modeled on $\mathbb{R}^n / \langle R \rangle$ for a reflection R , and the interior of each ridge ϵ is locally modeled on $(\mathbb{R}^n, D_{m_\epsilon}, \phi)$ for some chart ϕ where the dihedral group D_{m_ϵ} action is given by two reflections. The ridge ϵ is said to be of *order* m_ϵ . Every remaining point of P' in

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the faces of P has a neighborhood modeled as above with a finite *Coxeter group*, ie a group having a group presentation

$$\langle r_i \mid (r_i r_j)^{m_{ij}} \rangle \quad (i, j \in \mathbb{I})$$

where \mathbb{I} is a set, $m_{ii} = 1$ for all $i \in \mathbb{I}$, and $m_{ij} \in \{2, 3, \dots, +\infty\}$ is symmetric. Note that $m_{ij} = +\infty$ means no relations on r_i and r_j . The (orbifold) fundamental group of a Coxeter orbifold is isomorphic to a Coxeter group.

Given a Lie group G acting on a manifold X transitively, a (G, X) -structure on an orbifold Q is given by a maximal atlas of charts (U, H, ϕ) where U is an open subset of X and H is a finite subgroup of G and ϕ is a map $U \rightarrow \phi(U)$ inducing the quotient map $U \rightarrow U/H$.

In particular, we study real projective structures and hyperbolic structures on a Coxeter n -orbifold \hat{P} . A *real projective structure* on \hat{P} is a (G, X) -structure on \hat{P} with

$$G = \mathbf{SL}_{n+1}^{\pm}(\mathbb{R}) \quad \text{and} \quad X = \mathbb{S}^n$$

where the *projective sphere* \mathbb{S}^n is the set of rays in \mathbb{R}^{n+1} from the origin and $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ is the Lie group acting by projective transformations on \mathbb{S}^n , ie

$$\mathbf{SL}_{n+1}^{\pm}(\mathbb{R}) = \{A \in \mathbf{GL}_{n+1}(\mathbb{R}) : \det(A) = \pm 1\}.$$

We can represent hyperbolic structures on \hat{P} using Klein's projective model: the hyperbolic space \mathbb{H}^n is an open ball B in \mathbb{S}^n and the group of hyperbolic isometries is the subgroup $\mathbf{PO}(1, n)$ of $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ preserving B . Hence hyperbolic Coxeter orbifolds naturally have induced real projective structures.

Real projective structures have been studied by many mathematicians including Kuiper [26], Benzécri [8], Koszul [25], Goldman [21] and Benoist [4]. For the first time Kac and Vinberg [37] discovered hyperbolic Coxeter 2-orbifolds on which the induced real projective structures deform into a family of real projective structures that are not induced from hyperbolic structures. Johnson and Millson [24] constructed projective bending deformations of compact hyperbolic manifolds with embedded totally geodesic hypersurfaces. Cooper, Long and Thistlethwaite [15, 16] investigated whether the closed hyperbolic 3-manifolds of the Hodgson-Weeks census could be deformed. Heusener and Porti [23] provided infinite families of hyperbolic 3-manifolds which are locally projectively rigid, by means of Dehn filling. A survey on real projective structures is given in the article of Benoist [6].

The *deformation space* $\mathbb{D}(\hat{P})$ of *real projective structures* on the Coxeter orbifold \hat{P} is the space of real projective structures on \hat{P} up to equivalence relation given by isotopy in \hat{P} . The space has a natural C^1 -topology. We refer to Choi [12, 13] and Thurston [32, 33] for the details. A point p of $\mathbb{D}(\hat{P})$ gives a fundamental polyhedron P in \mathbb{S}^n , well defined up to projective transformations since the generating reflections determine P . The space of $p \in \mathbb{D}(\hat{P})$ giving a projectively fixed fundamental polyhedron P , which is called the *restricted deformation space* of real projective structures on \hat{P} , was studied on by Choi [13] and Choi, Hodgson and Lee [14].

In this paper we shall only consider deformation spaces $\mathbb{D}(\hat{P})$ of real projective structures on \hat{P} *without* the restrictions. Now we fix the dimension n to be so that $n \geq 3$. Let P be an n -dimensional hyperbolic convex polytope with dihedral angles submultiples of π ; we call P a *hyperbolic Coxeter n -polytope*. Then P naturally has a Coxeter orbifold structure \hat{P} by silvering the facets where each ridge is associated with the dihedral group $D_{n_{ij}}$ of order $2n_{ij}$ for when the ridge has the dihedral

angle equal to $\frac{\pi}{n_{ij}}$. The point t in $\mathbb{D}(\hat{P})$ is said to be *hyperbolic* if it is given by a hyperbolic structure on \hat{P} .

Definition 1.1. Let P be a hyperbolic Coxeter n -polytope, and let \hat{P} denote its Coxeter orbifold structure. Suppose that t is the corresponding hyperbolic point of $\mathbb{D}(\hat{P})$. We call a neighborhood of t in $\mathbb{D}(\hat{P})$ the *local deformation space* of \hat{P} . We say that \hat{P} is *projectively deformable*, or simply *deforms*, if the dimension of its local deformation space is positive. Conversely, we say that \hat{P} is *locally projectively rigid*, or *locally rigid*, if the dimension of its local deformation space is 0.

Definition 1.2. A Coxeter n -orbifold \hat{P} is *weakly orderable* if the facets of P can be ordered so that each facet contains at most n ridges of order 2 in facets of higher indices.

A convex n -polytope P in \mathbb{A}^n is called *simple* if exactly n facets meet at each vertex. The number of facets and ridges of P are denoted by f and e , respectively. We introduce an integer

$$\delta_P = e - nf + \frac{n(n+1)}{2}$$

which depends only on the polytope P and not on the orbifold structure. Barnette [2] showed for simple polytopes P that δ_P are non-negative. Observe that compact hyperbolic Coxeter n -polytopes are simple. Denote by e_+ the number of ridges of order ≥ 3 in \hat{P} .

Theorem 1.3. *Let P be a compact hyperbolic Coxeter n -polytope, and suppose that \hat{P} is the Coxeter orbifold arising from P . If the following two conditions are satisfied:*

- (C1) $\delta_P = 0$
- (C2) \hat{P} is weakly orderable

then a neighborhood of the hyperbolic point in $\mathbb{D}(\hat{P})$ is homeomorphic to a smooth manifold of dimension $e_+ - n$.

In other words, the compact hyperbolic Coxeter n -orbifold \hat{P} satisfying both (C1) and (C2) of Theorem 1.3 is projectively deformable if $e_+ > n$; otherwise, it is locally rigid.

Remark. Both (C1) and (C2) are necessary conditions; the details are given in Section 5.2 and Section 5.3.

If $n = 3$ and P is simple, then the Euler's formula implies that δ_P is always equal to 0.

Corollary 1.4. *Let P be a compact hyperbolic Coxeter 3-polytope, and suppose that \hat{P} is the Coxeter orbifold arising from P . If \hat{P} is weakly orderable, then a neighborhood of the hyperbolic point in $\mathbb{D}(\hat{P})$ is a smooth manifold of dimension $e_+ - 3$.*

A *truncation n -polytope* is a convex n -polytope obtained from the n -simplex by repeated truncations of vertices. It is well known that the condition (C2) of Theorem 1.3 holds for any truncation polytope. Moreover, if \hat{P} is any Coxeter orbifold arising from a truncation polytope P , then \hat{P} is weakly orderable as we can order facets by giving lower orders to all new facets arising from each step of the truncation process from the existing ones.

Corollary 1.5. *Let P be a compact hyperbolic Coxeter n -polytope, and suppose that \hat{P} is the Coxeter orbifold arising from P . If P is a truncation polytope, then a neighborhood of the hyperbolic point in $\mathbb{D}(\hat{P})$ is a smooth manifold of dimension $e_+ - n$.*

Marquis [28] used the word *ecimahedron* in place of *truncation 3-polytope* and showed that if \hat{P} is the Coxeter 3-orbifold arising from a compact hyperbolic Coxeter ecimahedron P then $\mathbb{D}(\hat{P})$ is diffeomorphic to \mathbb{R}^{e_+-3} .

Remark. Let P be a simple n -polytope, and suppose that \hat{P} is a Coxeter orbifold arising from P . As shown in Brøndsted [10, §19], for $n \geq 4$, P is a truncation n -polytope if and only if $\delta_P = 0$. Hence for $n \geq 4$, P is a truncation n -polytope if and only if \hat{P} satisfies both (C1) and (C2).

The remainder of this paper is organized as follows.

Section 2 reviews some known facts. In Section 2.1 we describe Vinberg's results giving the general conditions satisfied by real projective reflection groups. In fact, the work of Vinberg is central to the theory of real projective structures on Coxeter orbifolds. In Section 2.2 we recall Andreev's theorem characterizing the compact hyperbolic 3-polytope with dihedral angles at most $\frac{\pi}{2}$.

Section 3 gives various descriptions of the deformation space of real projective structures on a Coxeter orbifold \hat{P} . In Section 3.1 we introduce a space of representations of the orbifold fundamental group $\pi_1(\hat{P})$ and show that this representation space can be identified with the deformation space of real projective structures. In Section 3.2 we introduce the solution space of some polynomial equations given by Vinberg and a space of matrices satisfying certain conditions also given by Vinberg and establish the equivalence of these spaces.

Section 4 discusses general facts concerning a neighborhood of a hyperbolic structure in the deformation space of real projective structures on a Coxeter n -orbifold. In Section 4.1 we study the Zariski tangent space of the solution space of a corresponding system of Vinberg's polynomial equations. In Section 4.2 we introduce the polynomial equations defining a hyperbolic structure, and in Section 4.3 we describe the Zariski tangent space of these polynomial equations. In Section 4.4 we compare two Zariski tangent space at a hyperbolic point, and in Section 4.5 we combine this with the Weil infinitesimal rigidity to prove Theorem 1.3.

Section 5 we provides several examples and counterexamples. In Section 5.1 we prove that almost all of the compact hyperbolic 3-orbifolds arising from some 3-polytopes are weakly orderable. In Section 5.2 – Section 5.3 we show that two assumptions in Theorem 1.3 are necessary.

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2. PRELIMINARY

This section reviews the basic background material used in this article.

In Section 2.1 we describe Vinberg’s results giving the conditions under which a compact n –dimensional Coxeter orbifold admits a real projective structure. In Section 2.2 we recall Andreev’s theorem which explains when a compact 3–dimensional Coxeter orbifold admits a hyperbolic structure.

2.1. Vinberg’s results. This subsection gives a summary of results from Vinberg’s article [35]. An alternative treatment is given in Benoist’s notes [7].

Let V be an $(n+1)$ –dimensional real vector space. The projective sphere \mathbb{S}^n is the space of rays in V and double covers the projective space $\mathbb{R}P^n$. The group $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ acts on \mathbb{S}^n faithfully in the standard manner. The elements of $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ are said to be the *projective automorphisms* of \mathbb{S}^n and $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ the *projective automorphism group* of \mathbb{S}^n . Denote by π the natural projection from $V \setminus \{0\}$ into \mathbb{S}^n . A subspace of \mathbb{S}^n is the image of a subspace of V without the origin. In particular, a 2–dimensional subspace of V corresponds to a great circle in \mathbb{S}^n and an n –dimensional subspace gives a great $(n-1)$ –sphere in \mathbb{S}^n . Further, a component of the complement of a great $(n-1)$ –sphere can be identified with an affine n –space. We call this an *affine patch* of \mathbb{S}^n .

A *reflection* R is an element of order 2 of $\mathbf{GL}_{n+1}(\mathbb{R})$ which is the identity on a hyperplane U . All reflections are of the form

$$R = I_V - \alpha \otimes b$$

for some linear functional $\alpha \in V^*$ and a vector $b \in V$ with $\alpha(b) = 2$ and are in $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$. Observe that the kernel of α is the subspace U of fixed points of R and b is the *reflection vector*, ie an eigenvector corresponding to the eigenvalue -1 . Hence a reflection has a subspace of codimension-one as the set of fixed point and the point corresponding to the reflection vector is sent to its antipode. A *rotation* is an element of $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ which is the identity on a subspace of codimension two and is conjugate to a matrix $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ in a suitable supplementary basis. The real number θ is the *angle* of the rotation.

A subset Ω of \mathbb{S}^n is *convex* if its intersection with any great circle is connected. Moreover, it is *properly convex* if, in addition, its closure $\overline{\Omega}$ does not contain a pair of antipodal points. A subset P of \mathbb{S}^n is a *convex n –polytope* in \mathbb{S}^n if P is a convex n –polytope in an affine patch of \mathbb{S}^n . Observe that a convex n –polytope in \mathbb{S}^n is properly convex. As a matter of notation, given a convex n –polytope P in \mathbb{S}^n , $\text{cone}(P)$ will denote the convex polyhedral cone $\pi^{-1}(P) \cup \{0\}$ in V .

Let P be a convex n –polytope in \mathbb{S}^n , and for each facet F_i of P , take a linear functional α_i for F_i and choose a reflection $R_i = I_V - \alpha_i \otimes b_i$ with $\alpha_i(b_i) = 2$ which fixes F_i . By making a suitable choice of signs, we assume that P is defined by the inequalities

$$\alpha_i \geq 0, \quad i \in \mathbb{I} = \{1, \dots, f\}.$$

The group $\Gamma \subset \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ generated by all these reflections R_i is called a (*real*) *projective Coxeter group* if

$$\gamma P^{\circ} \cap P^{\circ} = \emptyset \quad \text{for every } \gamma \in \Gamma \setminus \{1\}$$

where P° is the interior of P . Note that Vinberg [35] used the word *linear Coxeter group* in place of *projective Coxeter group*. The $f \times f$ matrix $A = (a_{ij})$, $a_{ij} = \alpha_i(b_j)$, is called the *Cartan matrix* of Γ and P is called a *fundamental chamber* of Γ .

By [35, Theorem 1 and Proposition 6 and 17], the following conditions are necessary and sufficient for Γ to be a projective Coxeter group:

- (L1) $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.
- (L2) $a_{ii} = 2$; and for $i \neq j$, $a_{ij}a_{ji} \geq 4$ or $a_{ij}a_{ji} = 4\cos^2(\frac{\pi}{n_{ij}})$, n_{ij} an integer ≥ 2 .

In fact, if $a_{ij}a_{ji} = 4\cos^2(\frac{\pi}{n_{ij}})$ then the product R_iR_j is a rotation of angle $\frac{2\pi}{n_{ij}}$ and the group generated by two reflections R_i and R_j is the dihedral group $D_{n_{ij}}$. In particular, if $a_{ij} = a_{ji} = 0$ then R_iR_j is a rotation of angle $\frac{\pi}{2}$ and R_i and R_j generate a dihedral group of order 4, ie a Klein four group. If $a_{ij}a_{ji} \geq 4$ then R_i and R_j generate an infinite group and $n_{ij} = +\infty$.

For each reflection R_i , α_i and b_i are defined up to transformations

$$(2.1) \quad \alpha_i \mapsto d_i \alpha_i \text{ and } b_i \mapsto d_i^{-1} b_i \text{ with } d_i > 0.$$

Hence the Cartan matrix A of Γ is defined up to the action of a group of diagonal matrices with positive diagonal entries.

For any $x \in P$, let Γ_x denote the stabilizer subgroup of Γ of x . Define $P^f = \{x \in P : \Gamma_x \text{ is finite}\}$. By [35, Theorem 2], the following statements are true:

- $\Omega_\Gamma = \cup_{\gamma \in \Gamma} \gamma P$ is convex.
- Γ is a discrete subgroup of $\mathbf{SL}_{n+1}^\pm(\mathbb{R})$ preserving Ω_Γ° .
- $\Omega_\Gamma^\circ \cap P = P^f$, and is homeomorphic to $\Omega_\Gamma^\circ / \Gamma$.

Thus Ω_Γ° gives a convex open subset of the projective sphere \mathbb{S}^n , and $\Omega_\Gamma^\circ / \Gamma$ determines a *convex* real projective structure on the Coxeter n -orbifold \hat{P} associated with P . For example, let P be a hyperbolic Coxeter n -polytope of finite volume. Suppose that Γ is the discrete group generated by the reflections with respect to facets of P in the hyperbolic space \mathbb{H}^n . Then $\Omega_\Gamma^\circ = \mathbb{H}^n$ and $\Omega_\Gamma^\circ / \Gamma$ is a hyperbolic Coxeter n -orbifold.

A projective Coxeter group Γ is *elliptic*, *parabolic* and *hyperbolic* if Γ is derived from a discrete group generated by reflections in the sphere, Euclidean space and hyperbolic space, respectively, provided the following additional condition is satisfied: neither any proper plane in the hyperbolic space nor any point at infinity is Γ -invariant.

A matrix is called *indecomposable* if it cannot be represented as the direct sum of two matrices. Thus every matrix A decomposes into a direct sum of indecomposable matrices, which are called *components* of A . By Frobenius' theorem (See Gantmacher [19]), any indecomposable matrix A satisfying the condition (L1) has a real eigenvalue. It is said to be of *positive*, *zero* and *negative type* if the smallest real eigenvalue is positive, zero and negative, respectively. Denote by A^+ (resp. A^0 , A^-) the direct sum of its components of positive type (resp. zero type, negative type). Any matrix A satisfying the condition (L1) is represented as the direct sum of A^+ , A^0 and A^- .

Theorem 2.1. [35, Proposition 22 and 23] *Let Γ be a projective Coxeter group, and let A be the Cartan matrix of Γ .*

- Γ is elliptic $\Leftrightarrow A = A^+ \Leftrightarrow \Gamma$ is finite.
- Γ is parabolic $\Leftrightarrow A = A^0$ and $\text{rank}(A) = n$.

We shall consider only the case when $P = P^f$, which is equivalent to the assumption that $\Omega_\Gamma = \Omega_\Gamma^\circ$. We call Γ *perfect*. Observe that Γ is perfect $\Leftrightarrow \hat{P}$ is compact \Leftrightarrow the underlying space of \hat{P} equals P exactly.

Theorem 2.2. [35, Proposition 26] *Let Γ be a perfect projective Coxeter group, and let A be the Cartan matrix of Γ . Then exactly one of the following statements is true:*

- Γ is elliptic.
- Γ is parabolic.
- A is indecomposable and of negative type, and $\text{rank}(A) = \dim V$.

Lemma 2.3. [35, Lemma 15] *Let Γ be a projective Coxeter group. Assume that the Cartan matrix A of Γ is indecomposable and of negative type. Then Ω_Γ is properly convex.*

Let P be a convex n -polytope in \mathbb{S}^n and the polyhedral cone $K = \text{cone}(P)$ be given. The complex of K , denoted by $\mathfrak{F}K$, is the set of its (closed) faces, partially ordered by inclusion. Let K_1, \dots, K_f be the n -faces of K . For any face L of K , define $\sigma(L) = \{i \in \mathbb{I} : K_i \supset L\}$ and $\sigma(\mathfrak{F}K) = \{\sigma(L) \subset \mathbb{I} : L \in \mathfrak{F}K\}$. For any subset S of \mathbb{I} , the *standard subgroup* Γ_S of Γ is the subgroup generated by the reflection R_i , $i \in S$.

Theorem 2.4. [35, Theorem 7] *Let Γ be a perfect projective Coxeter group. Assume that P is a fundamental chamber of Γ and $K = \text{cone}(P)$. Then $S \in \sigma(\mathfrak{F}K)$ if and only if Γ_S is finite or $S = \mathbb{I}$.*

Note that the combinatorial structure of the fundamental chamber of a perfect projective Coxeter group Γ is completely determined by the abstract group structure of Γ .

For any subset S of \mathbb{I} , the *principal submatrix* A_S of A is the submatrix of A consisting of the entries a_{ij} for all $i, j \in S$. Denote by S^+ (resp. S^0 , S^-) the subset T of S such that $A_T = A_S^+$ (resp. A_S^0 , A_S^-).

Lemma 2.5. *Let Γ be a perfect projective Coxeter group, and let A be the Cartan matrix of Γ . If A has a principal submatrix of zero type, then Γ is parabolic.*

Proof. We refer to the proof of [35, Theorem 7]. Assume that Γ is not parabolic and $S = S^0$ for some $S \subset \mathbb{I}$. Define $Z(S) = \{i \in \mathbb{I} : a_{ij} = 0 \text{ for all } j \in S\}$ and $T = Z(S)^0$. Observe that $S \cup T = (S \cup T)^0$ and $Z(S \cup T)^0 = \emptyset$, and thus by [35, Theorem 4], $S \cup T \in \sigma(\mathfrak{F}K)$ with $K = \text{cone}(P)$. Theorem 2.1 and Theorem 2.2 show that $S \cup T \neq \mathbb{I}$. By Theorem 2.4, $\Gamma_{S \cup T}$ is finite. So is Γ_S , ie $S = S^+$ by Theorem 2.1. This is a contradiction. \square

Lemma 2.6. *Let Γ be a perfect projective Coxeter group, and let $A = (a_{ij})$ be the Cartan matrix of Γ . If Γ is not parabolic then $a_{ij}a_{ji} \neq 4$ for every $i \neq j$.*

Proof. If $a_{ij}a_{ji} = 4$ for some $i \neq j$ then the principal submatrix $\begin{bmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{bmatrix}$ of A is of zero type. By Lemma 2.5, Γ is parabolic. \square

Theorem 2.7. [35, Corollary 1] *Let A be an $f \times f$ matrix satisfying (L1) and (L2), and let $\text{rank}(A) = n + 1$. Suppose that A has no components of zero type. Then there exists a projective Coxeter group $\Gamma \subset \mathbf{SL}_{n+1}^\pm(\mathbb{R})$ with the Cartan matrix A . Furthermore, Γ is unique up to the conjugations in $\mathbf{SL}_{n+1}^\pm(\mathbb{R})$.*

2.2. Andreiev's theorem. For this subsection we refer to Andreiev [1] and Roeder, Hubbard and Dunbar [31].

A convex 3-polytope P is topologically a 3-ball B and the face structure of P gives B the structure of a cell complex whose k -cells correspond to the k -faces of P for $1 \leq k \leq 3$. The *boundary complex* ∂P of P is the subcomplex of P consisting of all proper faces. Let $(\partial P)^*$ be the dual complex of ∂P . A simple closed curve γ is called a k -circuit if it consists of k edges of $(\partial P)^*$ for some positive integer k . A circuit γ is *prismatic* if all of the endpoints of the edges of ∂P which γ meets are different.

Theorem 2.8. [1] *Suppose P is (the combinatorial type of) a simple 3-polytope, different from a tetrahedron, and a non-obtuse angle $\theta_{ij} \in (0, \frac{\pi}{2}]$ is given corresponding to each edge $F_{ij} = F_i \cap F_j$ of P , where F_i are the facets of P . Then the following conditions (A1)–(A4) are necessary and sufficient for the existence of a compact hyperbolic 3-polytope which realizes P with dihedral angle θ_{ij} at each edge F_{ij} .*

- (A1) *If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of P , then $\theta_{ij} + \theta_{jk} + \theta_{ki} > \pi$.*
- (A2) *If F_i, F_j and F_k form a prismatic 3-circuit, then $\theta_{ij} + \theta_{jk} + \theta_{ki} < \pi$.*
- (A3) *If F_i, F_j, F_k and F_l form a prismatic 4-circuit, then $\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} < 2\pi$.*
- (A4) *If P is a triangular prism with triangular faces F_1 and F_2 , then*

$$\theta_{13} + \theta_{14} + \theta_{15} + \theta_{23} + \theta_{24} + \theta_{25} < 3\pi.$$

Furthermore, this compact hyperbolic 3-polytope is unique up to hyperbolic isometries.

3. DEFORMATION SPACES OF REAL PROJECTIVE STRUCTURES

Through this section, we have three descriptions of the deformation space of real projective structures on a compact n -dimensional Coxeter orbifold \hat{P} , when \hat{P} admits a real projective structure but does not admit a spherical or Euclidean structure. In Section 3.1, we describe the deformation space in terms of representations from $\pi_1(\hat{P})$ into $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$. In Section 3.2, we describe this representation space in terms of polynomial equations and Cartan matrices, respectively; these are introduced by Vinberg.

3.1. Deformation spaces and the representation spaces. Let \hat{P} be a compact Coxeter n -orbifold, and let $\mathbb{D}(\hat{P})$ be the deformation space of real projective structures on \hat{P} . Assume that \hat{P} admits a real projective structure, but does not admit a spherical or Euclidean structure.

The $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ -action on $\text{Hom}(\pi_1(\hat{P}), \mathbf{SL}_{n+1}^{\pm}(\mathbb{R}))$ by conjugation is not faithful since $\pm I_V$ is in the kernel and

$$\text{Hom}(\pi_1(\hat{P}), \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})) / \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$$

is equivalent to

$$\text{Hom}(\pi_1(\hat{P}), \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})) / \mathbf{SL}_{n+1}(\mathbb{R})$$

thus, we will study the later space only.

Denote by $D_{\text{rep}}(\hat{P})$ the subspace corresponding to the irreducible representations $h: \pi_1(\hat{P}) \rightarrow \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ each of which acts on a properly convex open subset Ω of \mathbb{S}^n with the compact quotient orbifold $\Omega/h(\pi_1(\hat{P}))$.

Theorem 3.1. *Let \hat{P} be a compact Coxeter n -orbifold. Assume that \hat{P} admits a real projective structure, but does not admit a spherical or Euclidean structure.*

- *$D_{\text{rep}}(\hat{P})$ is a union of components of $\text{Hom}(\pi_1(\hat{P}), \mathbf{SL}_{n+1}^{\pm}(\mathbb{R}))$ where $\mathbf{SL}_{n+1}(\mathbb{R})$ acts properly and freely with a Hausdorff quotient space.*
- *The deformation space $\mathbb{D}(\hat{P})$ of real projective structures on the Coxeter orbifold \hat{P} is homeomorphic to*

$$D_{\text{rep}}(\hat{P})/\mathbf{SL}_{n+1}(\mathbb{R}).$$

- *For each element $h : \pi_1(\hat{P}) \rightarrow \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ of $D_{\text{rep}}(\hat{P})$, there exists a unique properly convex open subset Ω of \mathbb{S}^n up to the antipodal map \mathcal{A} so that $\Omega/h(\pi_1(\hat{P}))$ is diffeomorphic to \hat{P} .*

Proof. The (orbifold) fundamental group $\pi_1(\hat{P})$ of \hat{P} is an infinite, non-affine and irreducible Coxeter group by Theorem 2.2. Hence, by Qi [30, Theorem 1.1], the center of any finite index subgroup of $\pi_1(\hat{P})$ is trivial, and so by Benoist [4, Theorem 2.2] or [5, Theorem 1.1], $D_{\text{rep}}(\hat{P})$ is a closed subspace of $\text{Hom}(\pi_1(\hat{P}), \mathbf{SL}_{n+1}^{\pm}(\mathbb{R}))$. The openness was shown by Koszul [25]. By Lemma 3.2, the conjugation action by $\mathbf{SL}_{n+1}(\mathbb{R})$ is proper and free. Thus the first item is proved.

Let \hat{P} have a real projective structure. Denote by \tilde{P} the universal cover of \hat{P} . There exists an immersion $D : \tilde{P} \rightarrow \mathbb{S}^n$ equivariant with respect to a homomorphism $h : \pi_1(\hat{P}) \rightarrow \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$. Observe that \tilde{P} is tessellated with images of a fundamental polyhedron P . Theorem 2.2 and Lemma 2.3 imply that D sends \tilde{P} diffeomorphic to a properly convex open subset of \mathbb{S}^n and thus h is in $D_{\text{rep}}(\hat{P})$. By Choi [12, Corollary 1], there exists a local homeomorphism

$$hol : \mathbb{D}(\hat{P}) \rightarrow D_{\text{rep}}(\hat{P})/\mathbf{SL}_{n+1}(\mathbb{R}).$$

Suppose that Ω_1 and Ω_2 are convex open subsets of \mathbb{S}^n where $h(\pi_1(\hat{P}))$ acts for $h \in D_{\text{rep}}(\hat{P})$. If $\Omega_1 \cap \Omega_2 \neq \emptyset$ and they are not equal, then there exists a proper connected open subspace $\Omega_1 \cap \Omega_2$ of Ω_1 so that $(\Omega_1 \cap \Omega_2)/\Gamma$ is homotopy equivalent to Ω_1/Γ for an orientation-preserving finite-index torsion-free subgroup Γ of $h(\pi_1(\hat{P}))$. It is absurd for a proper-open submanifold to be homotopy equivalent to the closed manifold by the homology theory. Thus, we have $\Omega_1 = \Omega_2$ or $\Omega_1 \cap \Omega_2 = \emptyset$.

Let $\mathcal{A} : \mathbb{S}^n \rightarrow \mathbb{S}^n$ denote the antipodal map which conjugates $h(\pi_1(\hat{P}))$ to itself. Then $\Omega_2 = \mathcal{A}(\Omega_1)$ or $\Omega_2 \cap \mathcal{A}(\Omega_1) = \emptyset$ by the same reasoning.

Also, the closures $\overline{\Omega}_1$ and $\overline{\Omega}_2$ and their images under \mathcal{A} are mutually disjoint: Otherwise, their intersection is a $h(\pi_1(\hat{P}))$ -invariant convex subset of lower-dimension than n , and h is reducible.

Suppose that $\overline{\Omega}_2 \subset \mathbb{S}^n - \overline{\Omega}_1 - \mathcal{A}(\overline{\Omega}_1)$. There exists an element γ in $h(\pi_1(\hat{P}))$ that has an attracting fixed point x in the boundary of Ω_1 with the eigenvalue of x having a norm strictly greater than all other eigenvalues by Benoist [3]. Thus, acting by γ^m as $m \rightarrow \infty$, we obtain that $\overline{\Omega}_2 \cap \overline{\Omega}_1 \neq \emptyset$. This is a contradiction.

Therefore, we conclude $\Omega_2 = \Omega_1$ or $\Omega_2 = \mathcal{A}(\Omega_1)$. As \mathcal{A} is a projective automorphism, this proves the injectivity of hol .

The surjectivity follows since each element of $D_{\text{rep}}(\hat{P})$ divides a convex open subset of \mathbb{S}^n by definition: Consider $\hat{P}_h := \Omega/h(\pi_1(\hat{P}))$ for h in $\text{Hom}(\pi_1(\hat{P}), \mathbf{SL}_{n+1}^{\pm}(\mathbb{R}))$. By Charney and Davis [11], \hat{P}_h has the same partial ordering of faces by inclusion maps as \hat{P} since $h(\pi_1(\hat{P}))$ is isomorphic to $\pi_1(\hat{P})$; that is, they are combinatorially

equivalent by a homeomorphism. We can show that \hat{P}_h is diffeomorphic to \hat{P} by induction on codimensions of strata and removing their tubular neighborhoods.

The last item was proved while proving the second one. \square

Let $\mathbf{SO}(1, n)$ denote the orientation-preserving Lorentzian subgroup acting on V with a Lorentzian norm of signature $-1, 1, \dots, 1$. This is a Lie subgroup of $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$. Note that $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ -action by conjugations on $(\mathbf{SL}_{n+1}^{\pm}(\mathbb{R}))^f$ is not faithful. Thus, we act by $\mathbf{SL}_{n+1}(\mathbb{R})$. The following lemma is a generalization of Choi [13, Lemma 1].

Lemma 3.2. *Let \mathcal{U} denote the subspace of all elements (g_1, \dots, g_f) in $(\mathbf{SL}_{n+1}^{\pm}(\mathbb{R}))^f$ such that g_i 's generate a Zariski dense subgroup Z in a union of components of $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ or in a union of components of $\mathbf{PO}(1, n)$. Then the $\mathbf{SL}_{n+1}(\mathbb{R})$ -action on $(\mathbf{SL}_{n+1}^{\pm}(\mathbb{R}))^f$ by conjugation*

$$g \circ (g_1, \dots, g_f) = (gg_1g^{-1}, \dots, gg_fg^{-1})$$

is proper and fixed-point free on \mathcal{U} .

Proof. The proof for the properness directly generalize that of [13, Lemma 1] as the group Z is irreducible.

If there exists a fixed point of $g \in \mathbf{SL}_{n+1}(\mathbb{R})$, then g commutes with elements of Z and Z is not Zariski dense in union of components of $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$. If Z is Zariski dense in a union of components of $\mathbf{PO}(1, n)$, then an element commuting with elements of Z commutes with a cocompact discrete subgroup of a conjugate of $\mathbf{SO}(1, n)$. Then this element has to be trivial. Thus, there are no fixed points. \square

3.2. The reinterpretations of the deformation spaces as solution spaces.

Let V be an $(n+1)$ -dimensional real vector space. Denote by $M_{s \times t}(\mathbb{R})$ the set of $s \times t$ matrices with real entries. We will identify V and V^* with $M_{(n+1) \times 1}(\mathbb{R}) = \mathbb{R}^n$ and $M_{1 \times (n+1)}(\mathbb{R}) = (\mathbb{R}^n)^*$ respectively as follows: We choose the standard ordered basis $\{e_1, \dots, e_{n+1}\}$ of V . Let $\{e_1^*, \dots, e_{n+1}^*\}$ be its dual basis of V^* . If $\alpha_i = \alpha_{i,1}e_1^* + \dots + \alpha_{i,n+1}e_{n+1}^* \in V^*$ then α_i is identified with the $1 \times (n+1)$ matrix $(\alpha_{i,1}, \dots, \alpha_{i,n+1})$. Similarly, if $b_j = b_{j,1}e_1 + \dots + b_{j,n+1}e_{n+1} \in V$ then b_j is identified with the $(n+1) \times 1$ matrix $(b_{j,1}, \dots, b_{j,n+1})^t$, where the upper script t means the transpose of a matrix. Hence $\alpha_i(b_j) = \alpha_i b_j$ where the right-hand side is the scalar obtained as the matrix product of a $1 \times (n+1)$ matrix with a $(n+1) \times 1$ matrix. Denote by I_{n+1} the $(n+1) \times (n+1)$ identity matrix. The reflection R is equal to $I_{n+1} - b\alpha$ where $\alpha \in V^*$ and $b \in V$ with $\alpha b = 2$. Moreover, if g and h are linear transformations of V , then their composition $h \circ g$ is equal to the product hg of two matrices.

Let P be a convex n -polytope with f facets in \mathbb{S}^n , and let $\mathbb{I} = \{1, \dots, f\}$. Assume that P is given by a system of linear inequalities $\alpha_i \geq 0$ ($i \in \mathbb{I}$) where α_i are linear functionals in V^* . Suppose that b_i are reflection vectors with $\alpha_i b_i = 2$. Let R_i be the reflections defined by $R_i = I_{n+1} - b_i \alpha_i$ for all $i \in \mathbb{I}$, and let $\Gamma \subset \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ be

the group generated by the reflection R_i .

Define

$$\begin{aligned} E_1 &= \{(i, j) \in \mathbb{I} \times \mathbb{I} : i = j\}, \\ E_2 &= \{(i, j) \in \mathbb{I} \times \mathbb{I} : i < j, F_i \text{ and } F_j \text{ are adjacent in } P \text{ and } n_{ij} = 2\}, \\ E_3 &= \{(i, j) \in \mathbb{I} \times \mathbb{I} : i < j, F_i \text{ and } F_j \text{ are adjacent in } P \text{ and } n_{ij} \geq 3\} \text{ and} \\ E_4 &= \{(i, j) \in \mathbb{I} \times \mathbb{I} : i < j, F_i \text{ and } F_j \text{ are not adjacent in } P\}. \end{aligned}$$

Fix orders n_{ij} for the ridges of P . We consider the deformation space of real projective structures on the corresponding Coxeter orbifold \hat{P} . As in Section 3.1 we assume that \hat{P} admits a real projective structure, but does not admit a spherical or Euclidean structure. Vinberg's result and Lemma 2.6 lead us to solve the following system of polynomial equations:

- $a_{ii} = \alpha_i b_i = 2$ for $(i, i) \in E_1$.
- $a_{ij} = \alpha_i b_j = 0$ and $a_{ji} = \alpha_j(b_i) = 0$ for $(i, j) \in E_2$.
- $a_{ij} a_{ji} = \alpha_i b_j \alpha_j b_i = 4 \cos^2(\frac{\pi}{n_{ij}})$ for $(i, j) \in E_3$.

We call these polynomial equations *Vinberg equations*. The α_i 's and b_i 's are *variables*. Let N be the number of Vinberg equations and let $\Phi_{\hat{P}}: (V^*)^f \times V^f \rightarrow \mathbb{R}^N$ be the map given by

$$(\alpha_1, \dots, \alpha_f, b_1, \dots, b_f) \mapsto (\Phi_1, \dots, \Phi_N)$$

where $\{\Phi_k\}_{k=1}^N$ is the set of polynomials $a_{ii} - 2$, a_{ij} , a_{ji} , or $a_{ij} a_{ji} - 4 \cos^2(\frac{\pi}{n_{ij}})$ as in the Vinberg equations. Denote by e_2 the number of ridges of order 2. Observe that $N = f + e + e_2$. Define

$$\begin{aligned} \mathcal{U} &= \{(\alpha_1, \dots, \alpha_f, b_1, \dots, b_f) \in (V^*)^f \times V^f : \\ &\quad a_{ij} < 0 \text{ and } a_{ji} < 0 \text{ if } (i, j) \in E_3 \cup E_4, \text{ and } a_{ij} a_{ji} > 4 \text{ if } (i, j) \in E_4\}. \end{aligned}$$

We consider the solution set

$$\tilde{\mathbb{D}}(\hat{P}) := \Phi_{\hat{P}}^{-1}(0) \cap \mathcal{U}$$

which consists of elements of $(V^*)^f \times V^f$ such that the projective Coxeter group Γ generated by $R_i = I_{n+1} - b_i \alpha_i$ gives the quotient orbifold Ω_{Γ}/Γ which is isomorphic to \hat{P} . Denote by θ the action of $\tilde{G} = (\mathbb{R}_+)^f \times \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$ on $\tilde{\mathbb{D}}(\hat{P})$ given by

$$\begin{aligned} (3.1) \quad (d_1, \dots, d_f, g) \cdot (\alpha_1, \dots, \alpha_f, b_1, \dots, b_f) \\ = (d_1 \alpha_1 g^{-1}, \dots, d_f \alpha_f g^{-1}, d_1^{-1} g b_1, \dots, d_f^{-1} g b_f), \end{aligned}$$

where $d_i \in \mathbb{R}_+$ for all $i \in \mathbb{I}$ and $g \in \mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$. Applying the action θ on $\tilde{\mathbb{D}}(\hat{P})$, we have

$$I_{n+1} - (d_i^{-1} g b_i) (d_i \alpha_i g^{-1}) = g(I_{n+1} - b_i \alpha_i) g^{-1} = g R_i g^{-1} \text{ for all } i \in \mathbb{I}.$$

Hence the action θ on $\tilde{\mathbb{D}}(\hat{P})$ corresponds to the conjugation in $\mathbf{SL}_{n+1}^{\pm}(\mathbb{R})$.

Theorem 3.3. *Let \hat{P} be a compact Coxeter n -orbifold. Assume that \hat{P} admits a real projective structure, but does not admit a spherical or Euclidean structure.*

- $D_{\text{rep}}(\hat{P})$ is homeomorphic to $\tilde{\mathbb{D}}(\hat{P})/(\mathbb{R}_+)^f$.

- The deformation space $\mathbb{D}(\hat{P})$ of real projective structures on the Coxeter orbifold \hat{P} is homeomorphic to

$$\tilde{\mathbb{D}}(\hat{P})/\tilde{G} = D_{\text{rep}}(\hat{P})/\mathbf{SL}_{n+1}(\mathbb{R})$$

Proof. The first item follows since the representation is given by assigning the fixed points and reflection facets whose ambiguity is understood by Equation (3.1). The second item follows by Theorem 3.1 and the first item. \square

Let $\mathbb{PV}(\hat{P})$ denote the space of $f \times f$ matrix $A = (a_{ij})$ satisfying (L1) and (L2) with $\text{rank}(A) = n + 1$ and no component of zero type. We recall from Equation (2.1) that there is a diagonal group action $(\mathbb{R}_+)^f$ on $\mathbb{PV}(\hat{P})$ given by

$$(3.2) \quad (d_1, \dots, d_f) \circ (a_{ij}) = (d_i d_j^{-1} a_{ij}).$$

Corollary 3.4. *Let \hat{P} be a compact Coxeter n -orbifold. Assume that \hat{P} admits a real projective structure, but does not admit a spherical or Euclidean structure. There exists a one-to-one correspondence between*

$$\mathbb{D}(\hat{P}) \leftrightarrow D_{\text{rep}}(\hat{P})/\mathbf{SL}_{n+1}(\mathbb{R}) \leftrightarrow \tilde{\mathbb{D}}(\hat{P})/\tilde{G} \leftrightarrow \mathbb{PV}(\hat{P})/(\mathbb{R}_+)^f.$$

Proof. Theorem 3.1 and Theorem 3.3 give the first and second correspondence. The map from the forth one to the second one is given by Theorem 2.7. The map from the second one to the fourth one is given by going to the third one and taking $\alpha_i(b_j)$ as the entries of the Cartan matrices. Clearly, these are inverses of each other. \square

4. REAL PROJECTIVE STRUCTURES NEAR THE HYPERBOLIC STRUCTURE

The purpose of this section is to obtain the information of real projective structures near the hyperbolic structure in terms of Zariski tangent spaces.

Recall in the previous section that real projective structures in the deformation space of \hat{P} correspond to solutions to Vinberg equations. In Section 4.1 we study the Zariski tangent space to this solution space. In Section 4.2 we describe a space of hyperbolic structures of \hat{P} in terms of polynomial equations, which are called hyperbolic equations, and in Section 4.3 we study the Zariski tangent space to the solution space of the hyperbolic equations. In Section 4.4 we compare these two Zariski tangent spaces and combine this with the weak orderability of \hat{P} to prove Lemma 4.2. Finally, in Section 4.5, we prove Theorem 1.3 using Lemma 4.2 and Weil infinitesimal rigidity.

4.1. The Zariski tangent space to the Vinberg equations. As in Section 3.2, we have variables $\alpha_i \in V^* = (\mathbb{R}^{n+1})^*$ and $b_i \in V = \mathbb{R}^{n+1}$ for $i \in \mathbb{I} = \{1, \dots, f\}$ and the Vinberg equations of the following form:

- $\Phi_{ii} = \alpha_i b_i - 2 = 0$ for all $(i, i) \in E_1$.
- $\Phi_{ij}^{[1]} = \alpha_i b_j = 0$ and $\Phi_{ij}^{[2]} = \alpha_j b_i = 0$ for all $(i, j) \in E_2$.
- $\Phi_{ij} = \alpha_i b_j \alpha_j b_i - 4 \cos^2(\frac{\pi}{n_{ij}})$ for all $(i, j) \in E_3$.

Recall that N is the number of Vinberg equations, ie $N = f + e + e_2$. Let $\pi_i^{[1]}: (V^*)^f \times V^f \rightarrow V^*$ and $\pi_i^{[2]}: (V^*)^f \times V^f \rightarrow V$ denote the projection onto the i th and the $(f + i)$ th factor for all $i \in \mathbb{I}$, respectively. For each $(i, j) \in E_3$, the

derivative of Φ_{ij} at $p = (\alpha_1, \dots, \alpha_f, b_1, \dots, b_f)$, considered as a linear map, is given by:

$$\begin{aligned} D\Phi_{ij}(\dot{p}) &= a_{ji}\dot{\alpha}_i b_j + a_{ij}\dot{\alpha}_j b_i + a_{ij}\alpha_j \dot{b}_i + a_{ji}\alpha_i \dot{b}_j \\ &= a_{ji}\pi_i^{[1]}(\dot{p})b_j + a_{ij}\pi_j^{[1]}(\dot{p})b_i + a_{ij}\alpha_j\pi_i^{[2]}(\dot{p}) + a_{ji}\alpha_i\pi_j^{[2]}(\dot{p}) \end{aligned}$$

for $\dot{p} = (\dot{\alpha}_1, \dots, \dot{\alpha}_f, \dot{b}_1, \dots, \dot{b}_f) \in (V^*)^f \times V^f$. Similarly, for each $(i, i) \in E_1$,

$$D\Phi_{ii}(\dot{p}) = \pi_i^{[1]}(\dot{p})b_i + \alpha_i\pi_i^{[2]}(\dot{p}),$$

and for each $(i, j) \in E_2$,

$$D\Phi_{ij}^{[1]}(\dot{p}) = \pi_i^{[1]}(\dot{p})b_j + \alpha_i\pi_j^{[2]}(\dot{p}) \quad \text{and} \quad D\Phi_{ij}^{[2]}(\dot{p}) = \pi_j^{[1]}(\dot{p})b_i + \alpha_j\pi_i^{[2]}(\dot{p}).$$

More explicitly, combining Vinberg equations gives a function $\Phi_{\hat{P}} : V^f \times (V^*)^f \rightarrow \mathbb{R}^N$ and the rows of the $N \times 2(n+1)f$ Jacobian matrix $[D\Phi_{\hat{P}}]$ are made up of blocks, each consisting of $(n+1)$ entries:

For all $(i, i) \in E_1$,

$$\begin{aligned} [D\Phi_{ii}] &= (0, \dots, 0, b_{i,1}, \dots, b_{i,n+1}, 0, \dots, 0, \alpha_{i,1}, \dots, \alpha_{i,n+1}, 0, \dots, 0) \\ &= (0, \dots, 0, \underbrace{b_i^t}_{i\text{th block}}, 0, \dots, 0, \underbrace{\alpha_i}_{(f+i)\text{th block}}, 0, \dots, 0). \end{aligned}$$

For all $(i, j) \in E_2$,

$$\begin{aligned} [D\Phi_{ij}^{[1]}] &= (0, \dots, 0, \underbrace{b_j^t}_{i\text{th}}, 0, \dots, 0, \underbrace{0}_{j\text{th}}, 0, \dots, 0, \underbrace{0}_{(f+i)\text{th}}, 0, \dots, 0, \underbrace{\alpha_i}_{(f+j)\text{th}}, 0, \dots, 0) \\ [D\Phi_{ij}^{[2]}] &= (0, \dots, 0, \underbrace{0}_{i\text{th}}, 0, \dots, 0, \underbrace{b_i^t}_{j\text{th}}, 0, \dots, 0, \underbrace{\alpha_j}_{(f+i)\text{th}}, 0, \dots, 0, \underbrace{0}_{(f+j)\text{th}}, 0, \dots, 0). \end{aligned}$$

For all $(i, j) \in E_3$,

$$[D\Phi_{ij}] = (0, \dots, 0, \underbrace{a_{ji}b_j^t}_{i\text{th}}, 0, \dots, 0, \underbrace{a_{ij}b_i^t}_{j\text{th}}, 0, \dots, 0, \underbrace{a_{ij}\alpha_j}_{(f+i)\text{th}}, 0, \dots, 0, \underbrace{a_{ji}\alpha_i}_{(f+j)\text{th}}, 0, \dots, 0).$$

Suppose that p is a point of $\Phi_{\hat{P}}^{-1}(0)$. Then the *Zariski tangent space at p* is the kernel of the Jacobian matrix $[D\Phi_{\hat{P}}]$ evaluated at p .

4.2. The hyperbolic equations. Let V be an $(n+1)$ -dimensional real vector space with coordinates x_1, \dots, x_{n+1} , and let P be a Coxeter n -polytope in Klein's projective model of the n -dimensional hyperbolic space \mathbb{H}^n with facets F_i for $i \in \mathbb{I} = \{1, 2, \dots, f\}$. Denote by $\nu_i \in V$ the outward unit normal to F_i with respect to the Lorentzian inner product on V , defined by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + \dots + x_{n+1} y_{n+1}.$$

Then P is defined by the system of linear inequalities

$$\langle \nu_i, x \rangle \geq 0 \text{ for all } i \in \mathbb{I} \quad \text{and} \quad x_1 = 1.$$

Now the problem of constructing a hyperbolic Coxeter n -polytope P with prescribed dihedral angles $\frac{\pi}{n_{ij}}$ can be expressed as the problem of finding a solution to the following equations:

$$(4.1) \quad \begin{aligned} \langle \nu_i, \nu_i \rangle &= 1 \quad \text{for all } i \in \mathbb{I}, \\ \langle \nu_i, \nu_j \rangle &= -\cos\left(\frac{\pi}{n_{ij}}\right) \quad \text{if facets } F_i \text{ and } F_j \text{ are adjacent in } P. \end{aligned}$$

We call these equations *hyperbolic equations*. To compare this hyperbolic equations with Vinberg's equations, first note that P is defined by the system of linear inequalities

$$\alpha_i \geq 0 \text{ for all } i \in \mathbb{I} \quad \text{and} \quad x_1 = 1,$$

where the linear functional $\alpha_i \in V^*$ is dual to $2\nu_i$ under the Lorentzian inner product. In other words, $\alpha_i(v) = 2\langle \nu_i, v \rangle$. The hyperbolic reflection in the facet F_i is given by

$$R_i(v) = v - 2\langle \nu_i, v \rangle \nu_i = v - \alpha_i(v) b_i$$

where the reflection vector is $b_i = \nu_i$. Thus taking $\alpha_i = 2\langle \nu_i, \cdot \rangle$ and $b_i = \nu_i$ gives a *hyperbolic point* t in $\Phi_P^{-1}(0)$ corresponding to the hyperbolic structure on P :

if facets F_i and F_j are adjacent in P then

$$a_{ij} = \alpha_i(b_j) = 2\langle \nu_i, \nu_j \rangle = -2 \cos(\frac{\pi}{n_{ij}})$$

and thus

$$a_{ii} = 2\langle \nu_i, \nu_i \rangle = 2 \quad \text{for all } (i, i) \in E_1$$

$$a_{ij} = 0 \quad \text{and} \quad a_{ji} = 0 \quad \text{for all } (i, j) \in E_2$$

$$a_{ij}a_{ji} = 4 \cos^2(\frac{\pi}{n_{ij}}) \quad \text{for all } (i, j) \in E_3.$$

4.3. The Zariski tangent space to the hyperbolic equations. As in Section 4.2, we assume P is a compact hyperbolic Coxeter n -polytope where the dihedral angle at each ridge e_{ij} equals $\frac{\pi}{n_{ij}}$ for an integer $n_{ij} \geq 2$. Constructing such a hyperbolic n -polytope P is the same as solving the system of hyperbolic equations (4.1) for the unit normals ν_i . Equivalently we can write these equations in terms of the reflection vectors $b_i = \nu_i$. This gives the following system of $m = f + e$ equations:

$$(4.2) \quad \begin{aligned} \Psi_{ii} &= 2\langle b_i, b_i \rangle - 2 = 0 \quad \text{for all } (i, i) \in E_1 \\ \Psi_{ij} &= 2\langle b_i, b_j \rangle + 2 \cos(\frac{\pi}{n_{ij}}) = 0 \quad \text{for all } (i, j) \in E_2 \cup E_3. \end{aligned}$$

Combining these gives a function $\Psi_P : V^f = \mathbb{R}^{(n+1)f} \rightarrow \mathbb{R}^m$ and $\Psi_P^{-1}(0)$ contains Coxeter n -polytopes in \mathbb{H}^n with the desired dihedral angles.

Now consider the derivative $D\Psi_P$ at a hyperbolic point t . If $\alpha_i = 2\langle \nu_i, \cdot \rangle$ are the linear functionals defining the facets of the hyperbolic Coxeter n -polytope then

$$D\Psi_{ij}(\dot{b}) = 2\langle \dot{b}_i, b_j \rangle + 2\langle b_i, \dot{b}_j \rangle = \alpha_j \dot{b}_i + \alpha_i \dot{b}_j.$$

When $i = j$ this becomes

$$D\Psi_{ii}(\dot{b}) = 2\alpha_i \dot{b}_i.$$

Equivalently, the rows of the $m \times (n+1)f$ Jacobian matrix $[D\Psi_P]$ are made up of blocks, each consisting of $(n+1)$ entries:

For each $(i, i) \in E_1$,

$$\begin{aligned} [D\Psi_{ii}] &= (0, \dots, 0, 2\alpha_{i,1}, \dots, 2\alpha_{i,n+1}, 0, \dots, 0) \\ &= (0, \dots, 0, \underbrace{2\alpha_i}_{i\text{th block}}, 0, \dots, 0) \end{aligned}$$

and for each $(i, j) \in E_2 \cup E_3$,

$$[D\Psi_{ij}] = (0, \dots, 0, \underbrace{\alpha_j}_{i\text{th block}}, 0, \dots, 0, \underbrace{\alpha_i}_{j\text{th block}}, 0, \dots, 0).$$

Then the Zariski tangent space to $\Psi_P^{-1}(0)$ at t is $\ker D\Psi_P$.

Let V_1 denote the space of unit norm spacelike vectors in V . We consider $\Psi_{\hat{P}}^{-1}(0)$ as a subset of V_1 . We need to understand the properties of $\text{Hom}(\pi_1(\hat{P}), G)$ in terms of cocycles and coboundaries. See Goldman [20, Section 1] for an exposition here.

Proposition 4.1. *Let \hat{P} have a hyperbolic structure with the holonomy homomorphism $h: \pi_1(\hat{P}) \rightarrow \mathbf{PO}(1, n)$, and let (v_1, \dots, v_f) be the corresponding vectors in V_1^f normal to the facets of \hat{P} . Denote by $\mathcal{I}: V_1^f \rightarrow \mathbf{PO}(1, n)^f$ the map given by sending each normal vector to a corresponding reflection. Then the following holds:*

- *An open neighborhood in $\Psi_{\hat{P}}^{-1}(0)$ of (v_1, \dots, v_f) is identified with one in $\text{Hom}(\pi_1(\hat{P}), \mathbf{PO}(1, n))$ of h .*
- *$\Psi_{\hat{P}}^{-1}(0)$ is locally identical with an orbit space of h by the conjugation action $\mathbf{PO}(1, n)$ in $\text{Hom}(\pi_1(\hat{P}), \mathbf{PO}(1, n))$ and is a smooth manifold of dimension $\frac{n(n+1)}{2}$ near h , and $\dim B^1(\pi_1(\hat{P}), \mathfrak{so}(1, n)_{Ad_h}) = \frac{n(n+1)}{2}$ where $B^1(\pi_1(\hat{P}), \mathfrak{so}(1, n)_{Ad_h})$ is the space of coboundaries at h .*
- *There is an induced identification map*

$$D\mathcal{I}: \ker D\Psi_{\hat{P}} \rightarrow Z^1(\pi_1(\hat{P}), \mathfrak{so}(1, n)_{Ad_h}) = B^1(\pi_1(\hat{P}), \mathfrak{so}(1, n)_{Ad_h})$$

where $Z^1(\pi_1(\hat{P}), \mathfrak{so}(1, n)_{Ad_h})$ is the space of cocycles, ie the tangent space to $\text{Hom}(\pi_1(\hat{P}), \mathbf{PO}(1, n))$ at h .

Proof. Each point of $\Psi_{\hat{P}}^{-1}(0)$ correspond to a f -tuple of (R_1, \dots, R_f) reflections in $\mathbf{PO}(1, n)$ satisfying relations $R_i^2 = I$ and $R_i R_j$ is a rotation of angle $\frac{2\pi}{n_{ij}}$ where $i \neq j$ and $n_{ij} \neq +\infty$. This is locally one-to-one correspondence from an open neighborhood of $\Psi_{\hat{P}}^{-1}(0)$ to one of $\text{Hom}(\pi_1(\hat{P}), \mathbf{PO}(1, n))$ containing h . The first item follows by the fact that Equation (4.1) and Equation (4.2) express the same facts.

Since h is discrete and faithful and $h(\pi_1(\hat{P}))$ is Zariski dense and irreducible in $\mathbf{PO}(1, n)$, the orbit of h under the action of $\mathbf{PO}(1, n)$ is locally faithful and hence of dimension $\frac{n(n+1)}{2}$.

Let P be the fundamental domain of \hat{P} in the Klein model B of \mathbb{H}^n . A sufficiently nearby solution in $\Psi_{\hat{P}}^{-1}(0)$ to (v_1, \dots, v_f) corresponds to a polyhedron P' bounded by hyperplanes meeting at angles $\frac{\pi}{n_{ij}}$ where P' is near P in geometric distance. By the Poincaré polyhedron theorem, it follows that reflections with respect to facets of P' generate a discrete group Γ' isomorphic to $\pi_1(\hat{P})$ since the angle conditions do not change and hence has the same relations.

Denote by \hat{P}' the Coxeter orbifold B/Γ' . Then \hat{P}' is homeomorphic to \hat{P} by Charney and Davis [11] since $\pi_1(\hat{P}')$ is isomorphic to $\pi_1(\hat{P})$. We find a discrete faithful representation $h': \pi_1(\hat{P}') \rightarrow \mathbf{PO}(1, n)$. The Mostow rigidity [29] implies that \hat{P}' is isometric with \hat{P} and Γ' is conjugate to $\pi_1(\hat{P})$. Since $\pi_1(\hat{P}')$ can be identified with $\pi_1(\hat{P})$, we obtain that h' is conjugate to h by an element of $\mathbf{PO}(1, n)$. Thus a sufficiently nearby solution in $\Psi_{\hat{P}}^{-1}(0)$ to (v_1, \dots, v_f) all correspond to elements of orbit space of h under the conjugation action of $\mathbf{PO}(1, n)$. Also, the orbit space is locally smooth at h since $\mathbf{PO}(1, n)$ acts without fixed points as h is discrete faithful and irreducible. (In fact a neighborhood of h in $\text{Hom}(\pi_1(\hat{P}), \mathbf{PO}(1, n))$ is in the orbit space of h .)

We can consider $B^1(\pi_1(\hat{P}), so(1, n)_{Ad_h})$ as the tangent space of the orbit space of h by conjugation action of $\mathbf{PO}(1, n)$ at h . Thus, the second item follows.

For the last item, since $D\mathcal{I}$ is a derivative of \mathcal{I} at h , it sends the Zariski tangent space of $\Psi_{\hat{P}}^{-1}(0)$ to the Zariski tangent space $Z^1(\pi_1(\hat{P}), so(1, n)_{Ad_h})$ of $\text{Hom}(\pi_1(\hat{P}), \mathbf{PO}(1, n))$ at h . Also, by Weil infinitesimal rigidity [38], we have $H^1(\pi_1(\hat{P}), so(1, n)_{Ad_h}) = 0$. \square

4.4. Main Lemma.

Lemma 4.2. *Let P be a compact hyperbolic Coxeter n -polytope, and suppose that \hat{P} is the Coxeter orbifold arising from P . If \hat{P} is weakly orderable, then*

$$\text{rank } D\Phi_{\hat{P}} = \text{rank } D\Psi_{\hat{P}} + e_2 \text{ at the hyperbolic point}$$

where e_2 is the number of ridges of order 2.

Proof. Since \hat{P} is weakly orderable, the facets of P can be ordered so that each facet contains at most n ridges of order 2 in facets of higher indices. Let F_k be the k th facet, and let q be the largest index such that F_q contains ridges of order 2 in a facet of higher index. Define

$$\mathbb{I}(k) = \{i \in \mathbb{I} : i > k \text{ and } F_i \cap F_k \text{ is a ridge of order 2}\} \text{ and } i(k) = |\mathbb{I}(k)|.$$

We may enumerate

$$\mathbb{I}(k) = \{\mathbb{I}(k, 1), \dots, \mathbb{I}(k, i(k))\}$$

such that if $s < t$ then $\mathbb{I}(k, s) < \mathbb{I}(k, t)$. That is,

$$\begin{aligned} \mathbb{I}(1) &= \{\mathbb{I}(1, 1) < \mathbb{I}(1, 2) < \dots < \mathbb{I}(1, i(1))\} \\ \mathbb{I}(2) &= \{\mathbb{I}(2, 1) < \mathbb{I}(2, 2) < \dots < \mathbb{I}(2, i(2))\} \\ &\dots \\ \mathbb{I}(q) &= \{\mathbb{I}(q, 1) < \mathbb{I}(q, 2) < \dots < \mathbb{I}(q, i(q))\}. \end{aligned}$$

Then we have

$$\begin{aligned} E_2 &= \{(1, \mathbb{I}(1, 1)), (1, \mathbb{I}(1, 2)), \dots, (1, \mathbb{I}(1, i(1))), \\ &\quad (2, \mathbb{I}(2, 1)), (2, \mathbb{I}(2, 2)), \dots, (2, \mathbb{I}(2, i(2))), \\ &\quad \dots \\ &\quad (q, \mathbb{I}(q, 1)), (q, \mathbb{I}(q, 2)), \dots, (q, \mathbb{I}(q, i(q)))\} \end{aligned}$$

where $i(k) \leq n$. We note that

$$\sum_{k=1}^q i(k) = |E_2| = e_2.$$

Define the $1 \times (n+1)f$ matrices

$$\alpha_{[i]}^{[j]} = (0, \dots, 0, \underbrace{\alpha_i}_{j\text{th block}}, 0, \dots, 0) \quad \text{and} \quad b_{[i]}^{[j]} = (0, \dots, 0, \underbrace{b_i^t}_{j\text{th block}}, 0, \dots, 0).$$

Denote by J the $(n+1) \times (n+1)$ diagonal matrix with diagonal entries $-1, 1, \dots, 1$. We note that $\alpha_i = \nu_i^t J$, $b_i = \nu_i$ and $a_{ij} = a_{ij}$ at the hyperbolic point and the rows

of the $N \times 2(n+1)f$ Jacobian matrix $[D\Phi_{\hat{P}}]$ are represented as follows:

$$\begin{aligned} [D\Phi_{ii}] &= (b_{[i]}^{[i]}, \alpha_{[i]}^{[i]}) \text{ for all } (i, i) \in E_1 \\ [D\Phi_{ij}^{[1]}] &= (b_{[j]}^{[i]}, \alpha_{[i]}^{[j]}) \text{ and } [D\Phi_{ij}^{[2]}] = (b_{[i]}^{[j]}, \alpha_{[j]}^{[i]}) \text{ for all } (i, j) \in E_2 \\ [D\Phi_{ij}] &= (a_{ij}b_{[i]}^{[j]} + a_{ji}b_{[j]}^{[i]}, a_{ji}\alpha_{[i]}^{[j]} + a_{ij}\alpha_{[j]}^{[i]}) \text{ for all } (i, j) \in E_3. \end{aligned}$$

Now we use elementary row and column operations of $[D\Phi_{\hat{P}}]$ to obtain a matrix whose rank is easier to compute. Obviously, elementary row and column operations on a matrix are rank-preserving.

First, for each $(i, j) \in E_2$, add a row $[D\Phi_{ij}^{[1]}]$ of $[D\Phi_{\hat{P}}]$ to another row $[D\Phi_{ij}^{[2]}]$:

$$(b_{[i]}^{[j]}, \alpha_{[j]}^{[i]}) \rightarrow (b_{[i]}^{[j]} + b_{[j]}^{[i]}, \alpha_{[i]}^{[j]} + \alpha_{[j]}^{[i]}).$$

Second, for each $(i, j) \in E_3$, multiply a row $[D\Phi_{ij}]$ of $[D\Phi_{\hat{P}}]$ by a_{ij}^{-1} :

$$(a_{ij}b_{[i]}^{[j]} + a_{ji}b_{[j]}^{[i]}, a_{ji}\alpha_{[i]}^{[j]} + a_{ij}\alpha_{[j]}^{[i]}) \rightarrow (b_{[i]}^{[j]} + b_{[j]}^{[i]}, \alpha_{[i]}^{[j]} + \alpha_{[j]}^{[i]}).$$

Recall that for $(i, j) \in E_3$ each a_{ij} is non-zero and $a_{ij} = a_{ji}$ at the hyperbolic point in $\Phi^{-1}(0)$.

Third, for each $(i, i) \in E_1$, multiply a row $[D\Phi_{ii}]$ of $[D\Phi_{\hat{P}}]$ by 2:

$$(b_{[i]}^{[i]}, \alpha_{[i]}^{[i]}) \rightarrow (2b_{[i]}^{[i]}, 2\alpha_{[i]}^{[i]}).$$

Fourth, multiplying $(i(n+1) - n)$ th columns ($i \in \mathbb{I}$) of $[D\Phi_{\hat{P}}]$ by -1 produces the following result:

$$\begin{aligned} [D\Phi_{ij}^{[1]}] &\rightarrow (\alpha_{[j]}^{[i]}, \alpha_{[i]}^{[j]}) \text{ for all } (i, j) \in E_2 \\ [D\Phi_{ij}^{[2]}] &\rightarrow (\alpha_{[i]}^{[j]} + \alpha_{[j]}^{[i]}, \alpha_{[i]}^{[j]} + \alpha_{[j]}^{[i]}) \text{ for all } (i, j) \in E_2 \\ [D\Phi_{ij}] &\rightarrow (\alpha_{[i]}^{[j]} + \alpha_{[j]}^{[i]}, \alpha_{[i]}^{[j]} + \alpha_{[j]}^{[i]}) \text{ for all } (i, j) \in E_3 \\ [D\Phi_{ii}] &\rightarrow (2\alpha_{[i]}^{[i]}, 2\alpha_{[i]}^{[i]}) \text{ for all } (i, i) \in E_1. \end{aligned}$$

Similarly, the rows of the $m \times (n+1)f$ Jacobian matrix $[D\Psi_{\hat{P}}]$ are represented as follows:

$$\begin{aligned} [D\Psi_{ij}] &= \alpha_{[i]}^{[j]} + \alpha_{[j]}^{[i]} \text{ for all } (i, j) \in E_2 \cup E_3 \\ [D\Psi_{ii}] &= 2\alpha_{[i]}^{[i]} \text{ for all } (i, i) \in E_1. \end{aligned}$$

Comparing these two matrices, we observe that $[D\Phi_{\hat{P}}]$ was transformed into

$$[D\Phi_{\hat{P}}]^\nu := \left[\begin{array}{c|c} \begin{array}{c} \alpha_{[\mathbb{I}(1,1)]}^{[1]} \\ \dots \\ \alpha_{[\mathbb{I}(1,i(1))]}^{[1]} \\ \dots \end{array} & \begin{array}{c} \alpha_{[1]}^{[\mathbb{I}(1,1)]} \\ \dots \\ \alpha_{[1]}^{[\mathbb{I}(1,i(1))]} \\ \dots \end{array} \\ \hline \begin{array}{c} \alpha_{[\mathbb{I}(q,1)]}^{[q]} \\ \dots \\ \alpha_{[\mathbb{I}(q,i(q))]}^{[q]} \\ [D\Psi_{\hat{P}}] \end{array} & \begin{array}{c} \alpha_{[q]}^{[\mathbb{I}(q,1)]} \\ \dots \\ \alpha_{[q]}^{[\mathbb{I}(q,i(q))]} \\ [D\Psi_{\hat{P}}] \end{array} \end{array} \right]$$

Finally, using elementary column operations, we obtain

$$\left[\begin{array}{c|c} \alpha_{[\mathbb{I}(1,1)]}^{[1]} - \alpha_{[1]}^{[\mathbb{I}(1,1)]} & \alpha_{[1]}^{[\mathbb{I}(1,1)]} \\ \dots & \dots \\ \alpha_{[\mathbb{I}(1,i(1))]}^{[1]} - \alpha_{[1]}^{[\mathbb{I}(1,i(1))]} & \alpha_{[1]}^{[\mathbb{I}(1,i(1))]} \\ \dots & \dots \\ \dots & \dots \\ \alpha_{[\mathbb{I}(q,1)]}^{[q]} - \alpha_{[q]}^{[\mathbb{I}(q,1)]} & \alpha_{[q]}^{[\mathbb{I}(q,1)]} \\ \dots & \dots \\ \alpha_{[\mathbb{I}(q,i(q))]}^{[q]} - \alpha_{[q]}^{[\mathbb{I}(q,i(q))]} & \alpha_{[q]}^{[\mathbb{I}(q,i(q))]} \\ \hline 0 & [D\Psi_{\hat{P}}] \end{array} \right] = \left[\begin{array}{c|c|c|c|c} \alpha_{\mathbb{I}(1,1)} & * & * & * & * \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{\mathbb{I}(1,i(1))} & * & * & * & * \\ \hline 0 & \alpha_{\mathbb{I}(2,1)} & * & * & * \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \alpha_{\mathbb{I}(2,i(2))} & * & * & * \\ \hline \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \alpha_{\mathbb{I}(q,1)} & * \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \alpha_{\mathbb{I}(q,i(q))} & * \\ \hline 0 & \dots & 0 & 0 & [D\Psi_{\hat{P}}] \end{array} \right]$$

where 0's are zero matrices. Lemma 4.3 implies that for each $k \in \{1, 2, \dots, q\}$

$$\alpha_{\mathbb{I}(k,1)}, \alpha_{\mathbb{I}(k,2)}, \dots, \alpha_{\mathbb{I}(k,i(k))}$$

are linearly independent, ie all submatrices

$$S = \begin{bmatrix} \alpha_{\mathbb{I}(k,1)} \\ \alpha_{\mathbb{I}(k,2)} \\ \dots \\ \alpha_{\mathbb{I}(k,i(k))} \end{bmatrix}$$

are of full rank, establishing the result. \square

The following lemma, which is a generalization of Choi, Hodgson and Lee [14, Lemma 3], will be used again in Section 4.5.

Lemma 4.3. *Let P be a convex n -polytope in \mathbb{S}^n defined by linear inequalities $\alpha_i \geq 0$ where F_i be the facet of P determined by α_i . Suppose that the facets F_{i_1}, \dots, F_{i_n} are adjacent to the facet F_{i_0} . Then the $(n+1)$ linear functionals $\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_n}$ are linearly independent.*

Proof. Let π be the natural projection from $V \setminus \{0\}$ into \mathbb{S}^n , and let $S = \{i \in \mathbb{I} : F_i \text{ is adjacent to } F_{i_0}\}$. Define $W = \{v \in V : \alpha_{i_0}(v) = 0\}$ and $\mathbb{W} = \pi(W \setminus \{0\})$. Observe that W is an n -dimensional subspace of V and \mathbb{W} is a great $(n-1)$ -sphere. Denote by $\tilde{\alpha}$ the restricted linear functional $\alpha|_W : W \rightarrow \mathbb{R}$. Then the facet F_{i_0} of P is a convex $(n-1)$ -polytope in \mathbb{W} . Define

$$\tilde{Q} = \cap_{k=1}^n \{v \in W : \tilde{\alpha}_{i_k}(v) \geq 0\} \quad \text{and} \quad Q = \pi(\tilde{Q} \setminus \{0\}).$$

Since Q is an $(n-1)$ -dimensional simplex in \mathbb{W} , the linear functionals $\tilde{\alpha}_{i_k}$ ($k = 1, \dots, n$) are linearly independent. Consequently there is no non-zero vector $b \in V$ such that

$$\alpha_{i_0}(b) = \alpha_{i_1}(b) = \dots = \alpha_{i_n}(b) = 0$$

establishing the result. \square

Example 4.4. As an example, we use a compact hyperbolic tetrahedron to illustrate the method in the proof of Lemma 4.2. See Figure 1. Here, if an edge is labeled l the its dihedral angle is $\frac{\pi}{l}$.

Then
$$\boxed{\begin{array}{l} \mathbb{I}(1) = \{\mathbb{I}(1,1) = 3 < \mathbb{I}(1,2) = 4\} \\ \mathbb{I}(2) = \{\mathbb{I}(2,1) = 4\} \end{array} \mid \begin{array}{l} E_2 = \{(1,3), (1,4), \\ (2,4)\} \end{array}}$$

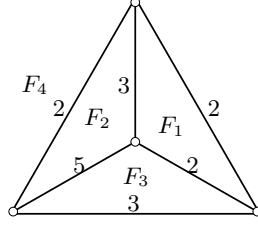


FIGURE 1. A compact hyperbolic tetrahedron

$$E_3 = \{(1, 2), (2, 3), (3, 4)\} \quad \text{and} \quad E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

and hence

$$[D\Phi_{\hat{P}}] = \begin{bmatrix} D\Phi_{ij}^{[1]} \text{ for each } (i, j) \in E_2 \\ D\Phi_{ij}^{[2]} \text{ for each } (i, j) \in E_2 \\ D\Phi_{ij} \text{ for each } (i, j) \in E_3 \\ D\Phi_{ii} \text{ for each } (i, i) \in E_1 \end{bmatrix} = \begin{bmatrix} D\Phi_{13}^{[1]} \\ D\Phi_{14}^{[1]} \\ D\Phi_{24}^{[1]} \\ D\Phi_{13}^{[2]} \\ D\Phi_{14}^{[2]} \\ D\Phi_{24}^{[2]} \\ D\Phi_{12} \\ D\Phi_{23} \\ D\Phi_{34} \\ D\Phi_{11} \\ D\Phi_{22} \\ D\Phi_{33} \\ D\Phi_{44} \end{bmatrix}$$

$$= \begin{bmatrix} b_3^t & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ b_4^t & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ 0 & b_4^t & 0 & 0 & 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & b_1^t & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1^t & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2^t & 0 & \alpha_4 & 0 & 0 \\ a_{21}b_2^t & a_{12}b_1^t & 0 & 0 & a_{12}\alpha_2 & a_{21}\alpha_1 & 0 & 0 \\ 0 & a_{32}b_3^t & a_{23}b_2^t & 0 & 0 & a_{23}\alpha_3 & a_{32}\alpha_2 & 0 \\ 0 & 0 & a_{43}b_4^t & a_{34}b_3^t & 0 & 0 & a_{34}\alpha_4 & a_{43}\alpha_3 \\ b_1^t & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & b_2^t & 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & b_3^t & 0 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & b_4^t & 0 & 0 & 0 & \alpha_4 \end{bmatrix}$$

where 0 is the 1×4 zero matrix.

First, for each $(i, j) \in E_2$, add a row $[D\Phi_{ij}^{[1]}]$ of $[D\Phi_{\hat{P}}]$ to another row $[D\Phi_{ij}^{[2]}]$:

$$\left[\begin{array}{cccc|cccc} b_3^t & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ b_4^t & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ \hline 0 & b_4^t & 0 & 0 & 0 & 0 & 0 & \alpha_2 \\ \hline b_3^t & 0 & b_1^t & 0 & \alpha_3 & 0 & \alpha_1 & 0 \\ b_4^t & 0 & 0 & b_1^t & \alpha_4 & 0 & 0 & \alpha_1 \\ 0 & b_4^t & 0 & b_2^t & 0 & \alpha_4 & 0 & \alpha_2 \\ a_{21}b_2^t & a_{12}b_1^t & 0 & 0 & a_{12}\alpha_2 & a_{21}\alpha_1 & 0 & 0 \\ 0 & a_{32}b_3^t & a_{23}b_2^t & 0 & 0 & a_{23}\alpha_3 & a_{32}\alpha_2 & 0 \\ 0 & 0 & a_{43}b_4^t & a_{34}b_3^t & 0 & 0 & a_{34}\alpha_4 & a_{43}\alpha_3 \\ b_1^t & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & b_2^t & 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & b_3^t & 0 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & b_4^t & 0 & 0 & 0 & \alpha_4 \end{array} \right].$$

Second, for each $(i, j) \in E_3$, multiply a row $[D\Phi_{ij}]$ of $[D\Phi_{\hat{P}}]$ by a_{ij}^{-1} :

$$\left[\begin{array}{cccc|cccc} b_3^t & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ b_4^t & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ \hline 0 & b_4^t & 0 & 0 & 0 & 0 & 0 & \alpha_2 \\ \hline b_3^t & 0 & b_1^t & 0 & \alpha_3 & 0 & \alpha_1 & 0 \\ b_4^t & 0 & 0 & b_1^t & \alpha_4 & 0 & 0 & \alpha_1 \\ 0 & b_4^t & 0 & b_2^t & 0 & \alpha_4 & 0 & \alpha_2 \\ b_2^t & b_1^t & 0 & 0 & \alpha_2 & \alpha_1 & 0 & 0 \\ 0 & b_3^t & b_2^t & 0 & 0 & \alpha_3 & \alpha_2 & 0 \\ 0 & 0 & b_4^t & b_3^t & 0 & 0 & \alpha_4 & \alpha_3 \\ b_1^t & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & b_2^t & 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & b_3^t & 0 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & b_4^t & 0 & 0 & 0 & \alpha_4 \end{array} \right].$$

Third, for each $(i, i) \in E_1$, multiply a row $[D\Phi_{ii}]$ of $[D\Phi_{\hat{P}}]$ by 2:

$$\left[\begin{array}{cccc|cccc} b_3^t & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ b_4^t & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ \hline 0 & b_4^t & 0 & 0 & 0 & 0 & 0 & \alpha_2 \\ \hline b_3^t & 0 & b_1^t & 0 & \alpha_3 & 0 & \alpha_1 & 0 \\ b_4^t & 0 & 0 & b_1^t & \alpha_4 & 0 & 0 & \alpha_1 \\ 0 & b_4^t & 0 & b_2^t & 0 & \alpha_4 & 0 & \alpha_2 \\ b_2^t & b_1^t & 0 & 0 & \alpha_2 & \alpha_1 & 0 & 0 \\ 0 & b_3^t & b_2^t & 0 & 0 & \alpha_3 & \alpha_2 & 0 \\ 0 & 0 & b_4^t & b_3^t & 0 & 0 & \alpha_4 & \alpha_3 \\ 2b_1^t & 0 & 0 & 0 & 2\alpha_1 & 0 & 0 & 0 \\ 0 & 2b_2^t & 0 & 0 & 0 & 2\alpha_2 & 0 & 0 \\ 0 & 0 & 2b_3^t & 0 & 0 & 0 & 2\alpha_3 & 0 \\ 0 & 0 & 0 & 2b_4^t & 0 & 0 & 0 & 2\alpha_4 \end{array} \right].$$

Fourth, multiplying $(4i - 3)$ th columns ($i \in \mathbb{I} = \{1, 2, 3, 4\}$) of $[D\Phi_{\hat{P}}]$ by -1 produces the following result:

$$\left[\begin{array}{cccc|cccc} \alpha_3 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ \hline 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & \alpha_2 \\ \hline \alpha_3 & 0 & \alpha_1 & 0 & \alpha_3 & 0 & \alpha_1 & 0 \\ \alpha_4 & 0 & 0 & \alpha_1 & \alpha_4 & 0 & 0 & \alpha_1 \\ 0 & \alpha_4 & 0 & \alpha_2 & 0 & \alpha_4 & 0 & \alpha_2 \\ \alpha_2 & \alpha_1 & 0 & 0 & \alpha_2 & \alpha_1 & 0 & 0 \\ 0 & \alpha_3 & \alpha_2 & 0 & 0 & \alpha_3 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 & \alpha_3 & 0 & 0 & \alpha_4 & \alpha_3 \\ 2\alpha_1 & 0 & 0 & 0 & 2\alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_2 & 0 & 0 & 0 & 2\alpha_2 & 0 & 0 \\ 0 & 0 & 2\alpha_3 & 0 & 0 & 0 & 2\alpha_3 & 0 \\ 0 & 0 & 0 & 2\alpha_4 & 0 & 0 & 0 & 2\alpha_4 \end{array} \right] = [D\Phi_{\hat{P}}]^\nu,$$

ie

$$[D\Phi_{\hat{P}}]^\nu = \left[\begin{array}{cccc|cccc} \alpha_3 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 \\ \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 \\ \hline 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & \alpha_2 \\ \hline & [D\Psi_{\hat{P}}] & & & & [D\Psi_{\hat{P}}] & & \end{array} \right].$$

Here we note that

$$[D\Psi_{\hat{P}}] = \left[\begin{array}{l} D\Psi_{ij} \text{ for each } (i, j) \in E_2 \\ \\ D\Psi_{ij} \text{ for each } (i, j) \in E_3 \\ \\ D\Psi_{ii} \text{ for each } (i, i) \in E_1 \end{array} \right] = \left[\begin{array}{l} D\Psi_{13} \\ D\Psi_{14} \\ D\Psi_{24} \\ D\Psi_{12} \\ D\Psi_{23} \\ D\Psi_{34} \\ D\Psi_{11} \\ D\Psi_{22} \\ D\Psi_{33} \\ D\Psi_{44} \end{array} \right] = \left[\begin{array}{cccc} \alpha_3 & 0 & \alpha_1 & 0 \\ \alpha_4 & 0 & 0 & \alpha_1 \\ 0 & \alpha_4 & 0 & \alpha_2 \\ \alpha_2 & \alpha_1 & 0 & 0 \\ 0 & \alpha_3 & \alpha_2 & 0 \\ 0 & 0 & \alpha_4 & \alpha_3 \\ 2\alpha_1 & 0 & 0 & 0 \\ 0 & 2\alpha_2 & 0 & 0 \\ 0 & 0 & 2\alpha_3 & 0 \\ 0 & 0 & 0 & 2\alpha_4 \end{array} \right].$$

Finally, using elementary column operations, we obtain

$$\left[\begin{array}{c|c|c|c|c} \alpha_3 & 0 & -\alpha_1 & 0 & 0 \ 0 \ \alpha_1 \ 0 \\ \alpha_4 & 0 & 0 & -\alpha_1 & 0 \ 0 \ 0 \ \alpha_1 \\ \hline 0 & \alpha_4 & 0 & -\alpha_2 & 0 \ 0 \ 0 \ \alpha_2 \\ \hline 0_{4 \times 10} & 0_{4 \times 10} & 0_{4 \times 10} & 0_{4 \times 10} & [D\Psi_{\hat{P}}] \end{array} \right]$$

where $0_{s \times t}$ is the $s \times t$ zero matrix.

4.5. Proof of Theorem 1.3.

Lemma 4.5. *Let \hat{P} be a compact Coxeter n -orbifold. Assume that \hat{P} admits a real projective structure, but does not admit a spherical or Euclidean structure. Define $\tilde{\mathbb{D}}(\hat{P})_r = \tilde{\mathbb{D}}(\hat{P}) \cap \mathcal{V} = \Phi_{\hat{P}}^{-1}(0) \cap (\mathcal{U} \cap \mathcal{V})$, where*

$$\mathcal{V} = \{p \in (V^*)^f \times V^f : \text{the differential of } \Phi_{\hat{P}} \text{ at } p \text{ is surjective}\}.$$

Then $\tilde{\mathbb{D}}(\hat{P})_r$ is a smooth manifold of dimension $2(n+1)f - N$.

Proof. The set $\mathcal{U} \cap \mathcal{V}$ is an open subset of $(V^*)^f \times V^f$ and the restriction to $\mathcal{U} \cap \mathcal{V}$ of the map $\Phi_{\hat{P}}$ is a submersion. Thus each level set of $\Phi_{\hat{P}}|_{\mathcal{U} \cap \mathcal{V}}$ is a embedded submanifold in $\mathcal{U} \cap \mathcal{V}$ whose codimension is equal to N . \square

We consider the action of \tilde{G} on the smooth manifold $\tilde{\mathbb{D}}(\hat{P})_r$ which is induced from the action θ on $\tilde{\mathbb{D}}(\hat{P})$ in Equation (3.1).

Lemma 4.6. *Let \hat{P} be a compact Coxeter n -orbifold. Assume that \hat{P} admits a real projective structure, but does not admit a spherical or Euclidean structure. Then \tilde{G} acts smoothly, freely and properly on a smooth manifold $\mathbb{D}(\hat{P})_r$. In particular, the orbit space $\mathbb{D}(\hat{P})_r/\tilde{G}$ is a smooth manifold of dimension $\dim \mathbb{D}(\hat{P})_r - \dim \tilde{G}$, and it is open in $\mathbb{D}(\hat{P})$.*

Proof. We show that \tilde{G} acts freely on a smooth manifold $\mathbb{D}(\hat{P})_r$. Suppose that

$$(d_1, \dots, d_f, g) \cdot (\alpha_1, \dots, \alpha_f, b_1, \dots, b_f) = (\alpha_1, \dots, \alpha_f, b_1, \dots, b_f)$$

where $d_1, \dots, d_f \in \mathbb{R}_+$ and $g \in \mathbf{SL}_{n+1}^\pm(\mathbb{R})$. That is,

$$d_i \alpha_i g^{-1} = \alpha_i \quad \text{and} \quad d_i^{-1} g b_i = b_i \quad \text{for all } i \in \mathbb{I}.$$

We have $d_i d_j^{-1} a_{ij} = a_{ij}$, hence $d_i = d_j$ if $\alpha_i(b_j) \neq 0$.

By Theorem 2.2, for any holonomy image group Γ of $\pi_1(\hat{P})$, the Cartan matrix of Γ is indecomposable. It follows that $d_1 = \dots = d_f$. Denote this like value by d . Choose $(n+1)$ linearly independent linear functionals $\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_n}$ as in Lemma 4.3. Let S be an invertible $(n+1) \times (n+1)$ matrix

$$S = \begin{bmatrix} \alpha_{i_0} \\ \alpha_{i_1} \\ \vdots \\ \alpha_{i_n} \end{bmatrix}$$

Then $d S g^{-1} = S$ and hence $d^{n+1} = \det(g) = 1$. Observe that $d = 1$ and $g = I_{n+1}$ establishing the result.

Next, we show that \tilde{G} acts properly on a smooth manifold $\mathbb{D}(\hat{P})_r$. Suppose that a sequence $\{p_k = (\alpha_{1,k}, \dots, \alpha_{f,k}, b_{1,k}, \dots, b_{f,k})\}$ converges to $(\alpha_1, \dots, \alpha_f, b_1, \dots, b_f)$ in $\mathbb{D}(\hat{P})_r$ and $\{q_k = (d_{1,k}, \dots, d_{f,k}, g_k)\}$ is a sequence in \tilde{G} such that $q_k \cdot p_k$ converges to $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_f, \tilde{b}_1, \dots, \tilde{b}_f)$ in $\mathbb{D}(\hat{P})_r$. That is,

$$(4.3) \quad d_{i,k} \alpha_{i,k} g_k^{-1} \rightarrow \tilde{\alpha}_i \quad \text{and} \quad d_{i,k}^{-1} g_k b_{i,k} \rightarrow \tilde{b}_i \quad \text{for all } i \in \mathbb{I}.$$

To complete the proof it suffices to establish that a subsequence of $\{q_k\}$ converges. We have $d_{i,k} d_{j,k}^{-1} \alpha_{i,k} b_{j,k} \rightarrow \tilde{\alpha}_i \tilde{b}_j$, hence $d_{i,k} d_{j,k}^{-1} \rightarrow \tilde{\alpha}_i \tilde{b}_j (\alpha_i b_j)^{-1}$ if $\alpha_i b_j \neq 0$. Moreover, $\tilde{\alpha}_i \tilde{b}_j$, $\tilde{\alpha}_j \tilde{b}_i$, $\alpha_i b_j$ and $\alpha_j b_i$ are negative if $(i, j) \notin E_1 \cup E_2$. Since the Cartan matrices $A = (a_{ij})$, $a_{ij} = \alpha_i b_j$, and $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = \tilde{\alpha}_i \tilde{b}_j$, are indecomposable it follows that for all $(i, j) \in \mathbb{I} \times \mathbb{I}$ there exist $c_{ij} > 0$ such that

$$(4.4) \quad d_{i,k} d_{j,k}^{-1} \rightarrow c_{ij}.$$

For any facet F_{i_0} of P , we can choose $(n+1)$ linearly independent linear functionals $\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_n}$ as in Lemma 4.3. Define $(n+1) \times (n+1)$ matrices

$$\tilde{S} = \begin{bmatrix} \tilde{\alpha}_{i_0} \\ \tilde{\alpha}_{i_1} \\ \vdots \\ \tilde{\alpha}_{i_n} \end{bmatrix}, \quad S = \begin{bmatrix} c_{i_0 i_0} \alpha_{i_0} \\ c_{i_1 i_0} \alpha_{i_1} \\ \vdots \\ c_{i_n i_0} \alpha_{i_n} \end{bmatrix} \quad \text{and} \quad S_k = \begin{bmatrix} d_{i_0,k} \alpha_{i_0,k} \\ d_{i_1,k} \alpha_{i_1,k} \\ \vdots \\ d_{i_n,k} \alpha_{i_n,k} \end{bmatrix}.$$

We note that \tilde{S} and S are invertible. By (4.3) and (4.4), $S_k g_k^{-1} \rightarrow \tilde{S}$ and $d_{i_0,k}^{-1} S_k \rightarrow S$, hence $d_{i_0,k} g_k^{-1} \rightarrow S^{-1} \tilde{S}$. Passing to a subsequence, we can assume that $\det g_k = 1$ or $\det g_k = -1$. Note that two ordered basis $(\tilde{\alpha}_{i_0}, \tilde{\alpha}_{i_1}, \dots, \tilde{\alpha}_{i_n})$ and $(\alpha_{i_0}, \alpha_{i_1}, \dots, \alpha_{i_n})$

consistently (resp. oppositely) oriented if $\det g_k = 1$ (resp. -1). Hence $d_{i_0,k}^{n+1}$ converges to a positive number

$$\det g_k \det(S^{-1}\tilde{S}).$$

Denote by d_{i_0} the $(n+1)$ th root of this limit. The (sub)sequences $\{d_{i_0,k}\}$ and $\{g_k\}$ converge to $\{d_{i_0}\}$ and $\{d_{i_0}\tilde{S}^{-1}S\}$, respectively. Since the index i_0 in \mathbb{I} can be arbitrarily chosen, the conclusion is immediate. \square

Theorem 4.7. *Let P be a compact hyperbolic Coxeter n -polytope, and suppose that \hat{P} is the Coxeter orbifold arising from P . Assume that P satisfies the condition (C1). Then the orbit space $\tilde{\mathbb{D}}(\hat{P})_r/\tilde{G}$ is a smooth manifold of dimension $e_+ - n$, and it is open in $\mathbb{D}(\hat{P})$.*

Proof. Let V be the number of Vinberg equations, and let e_2 be the number of ridges of order 2. Recall that $N = f + e + e_2$. We note that

$$\begin{aligned} \dim \tilde{\mathbb{D}}(\hat{P})_r - \dim \tilde{G} &= (2(n+1)f - N) - (f + (n+1)^2 - 1) \\ &= e_+ - n - 2\delta_P. \end{aligned}$$

Since $\delta_P = 0$, it follows that $\dim M - \dim \tilde{G} = e_+ - n$, and hence the conclusion is provided by Lemma 4.6. \square

Let $h : \pi_1(\hat{P}) \rightarrow \mathbf{PO}(1, n)$ is a discrete faithful representation corresponding to a hyperbolic structure on \hat{P} . By Weil infinitesimal rigidity [38],

$$H^1(\pi_1(\hat{P}), so(1, n)_{Ad_h}) = 0.$$

Since the adjoint action by $\mathbf{PO}(1, n)$ in the space of cocycles $Z^1(\pi_1(\hat{P}), so(1, n)_{Ad_h})$ is faithful, we obtain

$$\begin{aligned} \dim Z^1(\pi_1(\hat{P}), so(1, n)_{Ad_h}) &= \dim B^1(\pi_1(\hat{P}), so(1, n)_{Ad_h}) \\ &= \dim so(1, n) = \frac{n(n+1)}{2}. \end{aligned}$$

By Proposition 4.1, we have $\ker D\Psi_{\hat{P}} = Z^1(\pi_1(\hat{P}), so(1, n)_{Ad_h})$. Hence

$$\text{rank } D\Psi_{\hat{P}} = (n+1)f - \frac{n(n+1)}{2} = f + e - \delta_P$$

where $\delta_P = e - nf + \frac{n(n+1)}{2}$. Since $\delta_P = 0$ and \hat{P} is weakly orderable, it follows that $\text{rank } D\Phi_{\hat{P}} = \text{rank } D\Psi_{\hat{P}} + e_2 = f + e + e_2$ by Lemma 4.2, and so $D\Phi_{\hat{P}}$ is of full rank. Hence the differential of $\Phi_{\hat{P}}$ at the hyperbolic point t is surjective, ie $t \in \tilde{\mathbb{D}}(\hat{P})_r$. Finally Theorem 1.3 is provided by Theorem 4.7.

Remark. We use the following steps to confirm experimentally that the differentials of $\Phi_{\hat{P}}$ at the hyperbolic point are surjective for some compact hyperbolic Coxeter 3-orbifolds which are weakly orderable and have the combinatorial type of a dodecahedron. The main algorithm used for our computations is similar to the one in Choi, Hodgson and Lee [14, Section 4.5].

1. We tabulate some 3-dimensional compact hyperbolic Coxeter dodecahedra which are weakly orderable and satisfy the conditions of Andreev's theorem (A1)–(A4). In particular, we restrict the possible edge orders to obtain a manageable finite list. This gives us 5681 Coxeter orbifolds based on the dodecahedron.

2. We explicitly construct the 3-dimensional compact hyperbolic Coxeter dodecahedra obtained in step 1. To do this we obtain numerical values of the unit normals ν_i for the hyperbolic polyhedron in the list by deforming the dihedral angles of the dodecahedron `do13` in [14, Section 4.5]. By Vinberg [36, Theorem 2.1], we only need to show that the Gram matrix of the set of vectors ν_i is an indecomposable symmetric matrix of signature $(3, 1)$ with 1's along the diagonal and non-positive entries off it.
3. We compute the Jacobian matrix $D = [D\Phi_{\hat{P}}]$ of $\Phi_{\hat{P}}$ at the hyperbolic point obtained in step 2, and compare the number of rows of D with the rank of D to show that D is of full rank.

We use Matlab to check the conditions of Andreev's theorem in step 1, and we use Mathematica to compute the remaining steps of the algorithm. The detailed computations are available from the webpage [27].

5. EXAMPLES AND COUNTEREXAMPLES

Section 5 provides several examples of weakly orderable compact hyperbolic Coxeter 3-orbifolds and show that the both assumptions (C1) and (C2) of Theorem 1.3 are necessary conditions.

In Section 5.1 we prove that almost all of the compact hyperbolic 3-orbifold, which has no prismatic 3-circuit and has at most one prismatic 4-circuit, are weakly orderable. In Section 5.2–Section 5.3 we show that the local deformation spaces of real projective structures on some compact hyperbolic Coxeter n -orbifold \hat{P} which do not satisfy the assumption in Theorem 1.3 are not homeomorphic to a cell of dimension $e_+ - n$, where e_+ is the number of ridges of order ≥ 3 in \hat{P} .

5.1. Weakly orderable compact hyperbolic Coxeter 3-orbifolds. Every compact hyperbolic Coxeter 3-orbifold, which has the combinatorial type of a cube, is weakly orderable. However, if P is a dodecahedron then there exist a compact hyperbolic Coxeter 3-orbifold \hat{P} which is not weakly orderable. But, as shown in Theorem 5.1, almost all of the compact hyperbolic Coxeter 3-orbifolds, which have the combinatorial type of a dodecahedron, are weakly orderable.

We think that the following types of 3-polytopes form some portion of the set of all 3-polytopes admitting hyperbolic structures.

Theorem 5.1. *Let P be a simple 3-polytope. Suppose that P has no prismatic 3-circuit and has at most one prismatic 4-circuit. Then*

$$\lim_{d \rightarrow \infty} \frac{|\{\text{weakly orderable compact hyperbolic Coxeter 3-orbifolds } \hat{P}\}|}{|\{\text{compact hyperbolic Coxeter 3-orbifolds } \hat{P}\}|} = 1$$

where d is the maximum of edge orders of the Coxeter 3-orbifold \hat{P} .

Question 5.2. *Is Theorem 5.1 still true if we assume only that P is a simple 3-polytope?*

Before going to the proof of Theorem 5.1, we state the Tutte's theorem [34].

An 1-dimensional cell complex \mathbb{G} is called a *graph*. It consists of vertices (the 0-cells) to which edges (the 1-cells) are attached. The *degree* of a vertex in a graph is the number of edges with which it is incident. If all the vertices in a graph \mathbb{G} have degree d , \mathbb{G} is said to be *regular of degree d* .

A *subgraph* of \mathbb{G} is a graph having all of its vertices and edges in \mathbb{G} . A *spanning subgraph* of \mathbb{G} is a subgraph containing all the vertices of \mathbb{G} . A *factor* is a spanning subgraph which is regular of degree 1.

Theorem 5.3. [34] *Let \mathbb{G} be a finite graph, and let v be the number of vertices of \mathbb{G} . If \mathbb{G} is a d -connected graph with v even and is regular of degree d , then \mathbb{G} has a factor. Moreover, if, in addition, \mathfrak{e} is any edge of \mathbb{G} , then \mathbb{G} has a factor which contains \mathfrak{e} .*

Lemma 5.4. *Let P be a simple 3-polytope. Suppose that P has no prismatic 3-circuit and has at most one prismatic 4-circuit. Then there is a compact hyperbolic Coxeter 3-orbifold \hat{P} such that every vertex of \hat{P} is incident with two edges of order 2 and one edge of order 7.*

Proof. Assume that four facets F_i, F_j, F_k and F_l of P form a prismatic 4-circuit. Denote the edge $F_i \cap F_j$ by \mathfrak{e} . By Steinitz' theorem, the graph $\mathbb{G} = \mathbb{G}(P)$ of P is 3-connected (See Grünbaum [22, Chapter 13]). Since P is simple, \mathbb{G} is regular of degree 3 and the number of vertices is even. By Tutte's theorem, \mathbb{G} has a factor \mathbb{F} which contains \mathfrak{e} . (If P has no prismatic 4-circuit, then we choose an arbitrary factor \mathbb{F} of \mathbb{G} .) Every vertex of P is incident with two edges in $\mathbb{G} \setminus \mathbb{F}$ and one edge in \mathbb{F} . Observe that

$$\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{7} > \pi \quad \text{and} \quad \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{7} < 2\pi.$$

By Andreev's theorem in Section 2.2, there is a compact hyperbolic Coxeter 3-orbifold \hat{P} such that every edge in $\mathbb{G} \setminus \mathbb{F}$ (resp. \mathbb{F}) is of order 2 (resp. of order 7), corresponding to a dihedral angle $\frac{\pi}{2}$ (resp. $\frac{\pi}{7}$). \square

Let \mathbb{G} be a finite graph, and let L be a set. Denote by $E(\mathbb{G})$ the set of edges of \mathbb{G} . A function $\vartheta : E(\mathbb{G}) \rightarrow L$ is called an *edge-labeling function*, and we call a pair (\mathbb{G}, ϑ) an *edge-labeled graph*. An edge \mathfrak{e} is called an *l -edge* if $\vartheta(\mathfrak{e}) = l$.

In this section, we consider only the edge-labeled graph (\mathbb{G}, ϑ) satisfying the following conditions:

- (E1) \mathbb{G} is simple, planar and 3-connected, ie \mathbb{G} is the graph of a 3-polytope.
- (E2) \mathbb{G} is regular of degree 3.
- (E3) the set L of labels is $\{0, 1\}$.
- (E4) every vertex of \mathbb{G} is incident with three edges $\mathfrak{e}_1, \mathfrak{e}_2$ and \mathfrak{e}_3 such that

$$\vartheta(\mathfrak{e}_1) + \vartheta(\mathfrak{e}_2) + \vartheta(\mathfrak{e}_3) \equiv 1 \pmod{2}.$$

Let P be a convex 3-polytope in \mathbb{A}^3 , and let \mathbb{G} be the graph of P . The graph \mathbb{G} is embedded in the 2-dimensional sphere $S^2 = \partial P$. We call a facet of P a *face* of \mathbb{G} . The number of vertices, edges and faces of \mathbb{G} shall be denoted by v, e and f , respectively.

Lemma 5.5. *Let (\mathbb{G}, ϑ) be an edge-labeled graph satisfying the condition (E1)–(E4). Then there is a face F of \mathbb{G} such that the number of 0-edges of F is less than 4.*

Proof. Denote by e_2 the number of 0-edges. Observe that

- (E1) implies that $v - e + f = 2$
- (E2) implies that $2e = 3v$
- (E3) and (E4) imply that $2e_2 \leq 2v$.

By an elementary computation, we have

$$2e_2 \leq 4(f - 2) < 4f.$$

The conclusion is immediate. \square

We define a combinatorial operation on edge-labeled graphs satisfying the condition (E1)–(E4) (resp. on graphs satisfying the condition (E1)–(E2)): By *deleting* an edge $\mathfrak{e} = \mathfrak{a}\mathfrak{b}$ from an edge-labeled graph (\mathbb{G}, ϑ) (resp. \mathbb{G}) we mean that \mathfrak{e} is removed and the pairs of edges incident to \mathfrak{a} and \mathfrak{b} are amalgamated into two edges (see Figure 2). The possibility of applying the deleting operation to an edge \mathfrak{e} of (\mathbb{G}, ϑ) presupposes that two adjacent edges which are adjacent to an edge \mathfrak{e} have the same label. We note that the conditions (E2)–(E4) (resp. (E2)) are preserved under the edge-deleting operations.

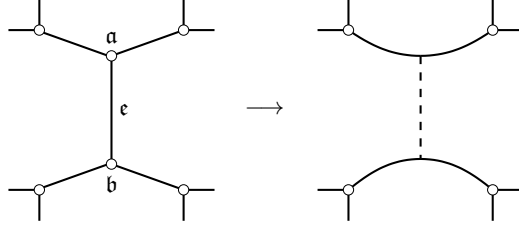


FIGURE 2. An edge-deleting operation

Let \mathbb{G} be a graph satisfying the condition (E1)–(E2). An edge \mathfrak{e} of \mathbb{G} is said to be *removable* when the graph obtained from \mathbb{G} by deleting the edge \mathfrak{e} remains to satisfy the condition (E1). An edge which is not removable is said to be *non-removable*. The following property of the set of non-removable edges was obtained by Fouquet and Thuillier [18].

Theorem 5.6. [18, Corollary 2.7] *Let \mathbb{G} be a graph with more than 6 edges satisfying the condition (E1)–(E2), and let C be a cycle of \mathbb{G} . Then C contains at least two removable edges.*

Lemma 5.7. *Let P be a compact hyperbolic 3-polytope, and let \hat{P} be the Coxeter 3-orbifold arising from P . Assume that every vertex of \hat{P} is incident with two edges of order 2 and one edge of order ≥ 7 . Then \hat{P} is weakly orderable.*

Proof. If P is a tetrahedron then P is always weakly orderable. We may assume that P is not tetrahedron. Denote by \mathbb{G} the graph of the 3-polytope P . Define the edge-labeling function ϑ by

$$\vartheta(\mathfrak{e}) = \begin{cases} 0 & \text{if the edge } \mathfrak{e} \text{ is of order 2,} \\ 1 & \text{if the edge } \mathfrak{e} \text{ is of order } \geq 7. \end{cases}$$

Then the edge-labeled graph (\mathbb{G}, ϑ) satisfies the conditions (E1)–(E4).

By Lemma 5.5 there is a face F of \mathbb{G} such that the number of 0-edges of F is less than 4. We call F the 1st facet F_1 of P . By Theorem 5.6 the cycle ∂F contains a removable edge \mathfrak{e} .

If $\vartheta(\mathfrak{e}) = 1$ then adjacent edges which are adjacent to \mathfrak{e} have the same label. Otherwise, we relabel all edges in the cycle ∂F to become edges of the opposite

label, and obtain the new labeling function ϑ' . Observe that $\vartheta'(\mathfrak{e}) = 1$ and the resulting edge-labeled graph (\mathbb{G}, ϑ') still satisfies the condition (E4).

Denote by F' the facet adjacent to F such that $F \cap F' = \mathfrak{e}$. We delete the edge \mathfrak{e} of (\mathbb{G}, ϑ') . Two adjacent facets F and F' are amalgamated into a facet F'' (See Figure 3).

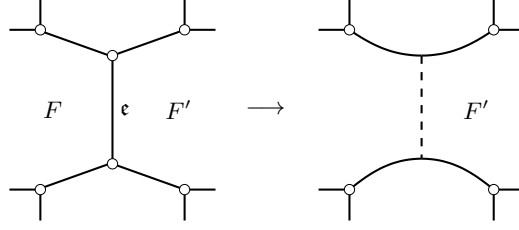


FIGURE 3. Amalgamating two adjacent facets into a facet

Now, the resulting edge-labeled graph $(\tilde{\mathbb{G}}, \tilde{\vartheta})$ has fewer edges. However, it still satisfies all of the condition (E1)–(E4). Again, find an face \tilde{F} of $\tilde{\mathbb{G}}$ such that the number of 0-edges of \tilde{F} is less than 4. We call \tilde{F} the 2nd facet F_2 of P . Continue the process in this manner.

Consequently the facet of \hat{P} can be ordered so that each facet contains at most 3 edges of order 2 in a facet of higher index, and hence \hat{P} is weakly orderable. In fact, we are done by induction on the number of edges. \square

Proof of Theorem 5.1. Let e be the number of edges of P , and let $p = \frac{1}{3}e$. Observe that $p \in \mathbb{N}$. Denote by $\mathcal{N}(d)$ the set of compact hyperbolic Coxeter orbifold \hat{P} such that all the edge orders of \hat{P} are less than or equal to d . For each integer $d \geq 7$ and $j \in \{0, 1, \dots, e\}$, we define

$$\mathcal{N}_\omega(d) = \{ \hat{P} \in \mathcal{N}(d) : \hat{P} \text{ is weakly orderable} \}$$

$$\mathcal{N}_j(d) = \{ \hat{P} \in \mathcal{N}(d) : \text{the number of edges of order } \geq 7 \text{ in } \hat{P} \text{ is equal to } j \}.$$

Assume that \hat{P} is a compact hyperbolic Coxeter 3-orbifold. By (A1) of Andreev's theorem in Section 2.2, if an edge \mathfrak{e} of \hat{P} is of order ≥ 7 then edges which are adjacent to \mathfrak{e} are of order 2. Therefore the maximum number of edges of order ≥ 7 in \hat{P} is equal to $p = \frac{1}{3}e$. In other words, $\mathcal{N}_j(d) = \emptyset$ for all $j > p$, and hence we have

$$|\mathcal{N}(d)| = \sum_{j=0}^p |\mathcal{N}_j(d)|.$$

Moreover, $\hat{P} \in \mathcal{N}_p(d)$ if and only if every vertex of \hat{P} is incident with two edges of order 2 and one edge of order ≥ 7 . Observe that for any fixed integers $l, m \geq 2$,

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1 \text{ for some integer } k \geq 7 \Leftrightarrow \frac{1}{k} + \frac{1}{l} + \frac{1}{m} > 1 \text{ for all integer } k \geq 7, \text{ and}$$

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1 \text{ for some integer } k \geq 7 \Leftrightarrow \frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1 \text{ for all integer } k \geq 7$$

(compare these inequalities to (A1) and (A2) of Andreev's theorem). Consequently for each $d \in \{7, 8, \dots\}$ we have

$$|\mathcal{N}_j(d)| = |\mathcal{N}_j(7)| \cdot (d-6)^j.$$

Lemma 5.4 and Lemma 5.7 imply that

$$\frac{|\mathcal{N}_\omega(d)|}{|\mathcal{N}(d)|} \geq \frac{|\mathcal{N}_p(d)|}{|\mathcal{N}(d)|} = \frac{|\mathcal{N}_p(7)| \cdot (d-6)^p}{\sum_{j=0}^p |\mathcal{N}_j(7)| \cdot (d-6)^j} \quad \text{and} \quad |\mathcal{N}_p(7)| \neq 0,$$

establishing the result. \square

Example 5.8. Let m be an integer ≥ 5 . By Löbell 3-polytope $L(m)$ we mean a 3-polytope which has $(2m+2)$ facets with upper and lower bases both being m -gons, and a lateral surface given by $2m$ pentagons, arranged similarly as in the dodecahedron. Figure 4 shows the case when $m = 6$.

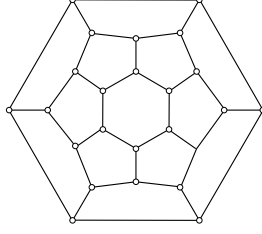


FIGURE 4. Löbell 3-polytope $L(6)$

For each $m \geq 5$, the Löbell 3-polytope $L(m)$ has no prismatic 3- or 4-circuits. By Theorem 5.1 almost all of the compact hyperbolic Coxeter 3-orbifold which has combinatorial type of $L(m)$ are weakly orderable.

5.2. The condition (C1) is necessary. In 1996, Esselmann [17] classified all the compact hyperbolic Coxeter polytopes the combinatorial type of which is the product of two simplicies of dimension greater than 1. We consider one of these hyperbolic polytopes. Let P be the compact hyperbolic Coxeter 4-polytope the combinatorial type of which is the product of two triangles and the Coxeter graph of which is shown in Figure 5. See Vinberg [36] or Bourbaki [9] for the definition of Coxeter graphs.

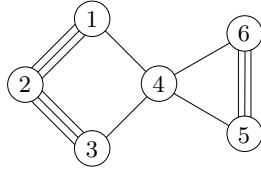


FIGURE 5. One of Esselmann's polytopes

Since the 4-polytope P has 6 facets and 15 ridges,

$$\delta_P = e - nf + \frac{n(n+1)}{2} = 1 \neq 0, \text{ ie } P \text{ does not satisfy the condition (C1).}$$

However the Coxeter orbifold \hat{P} arising from P is weakly orderable, ie \hat{P} satisfy the condition (C2).

We show that any neighborhood of the hyperbolic point in $\mathbb{D}(\hat{P})$ for the hyperbolic Coxeter orbifold \hat{P} is *not* a manifold.

Assume that Γ is a projective Coxeter group so that Ω_Γ/Γ is diffeomorphic to \hat{P} , and A is the Cartan matrix of Γ . We can change the Cartan matrix A by a diagonal action (see Equation (3.2)) uniquely so that

$$\begin{aligned} a_{12} = a_{21} &= -2 \cos(\frac{\pi}{5}), & a_{23} = a_{32} &= -2 \cos(\frac{\pi}{5}), & a_{34} = a_{43} &= -2 \cos(\frac{\pi}{3}) \\ a_{45} = a_{54} &= -2 \cos(\frac{\pi}{3}), & a_{56} = a_{65} &= -2 \cos(\frac{\pi}{5}). \end{aligned}$$

Define $x = -a_{14}$ and $y = -a_{46}$. The Cartan matrix $A = (a_{ij})$ of Γ is as follows:

$$A = \begin{bmatrix} 2 & -\frac{1+\sqrt{5}}{2} & 0 & -x & 0 & 0 \\ -\frac{1+\sqrt{5}}{2} & 2 & -\frac{1+\sqrt{5}}{2} & 0 & 0 & 0 \\ 0 & -\frac{1+\sqrt{5}}{2} & 2 & -1 & 0 & 0 \\ -x^{-1} & 0 & -1 & 2 & -1 & -y \\ 0 & 0 & 0 & -1 & 2 & -\frac{1+\sqrt{5}}{2} \\ 0 & 0 & 0 & -y^{-1} & -\frac{1+\sqrt{5}}{2} & 2 \end{bmatrix}.$$

Moreover, $\text{rank}(A) = 5$ if and only if $\det(A) = 0$. By simple calculation, we obtain

$$\det(A) = \frac{1}{2xy}(8x - (5 + \sqrt{5})y - (6 - 2\sqrt{5})xy - (5 + \sqrt{5})x^2y + 8xy^2) = 0.$$

Note that x and y are positive. By Corollary 3.4, the deformation space $\mathbb{D}(\hat{P})$ is homeomorphic to the solution space

$$\mathcal{S} = \{(x, y) \in (\mathbb{R}_+)^2 : 8x - (5 + \sqrt{5})y - (6 - 2\sqrt{5})xy - (5 + \sqrt{5})x^2y + 8xy^2 = 0\},$$

which is pictured in Figure 6. We note that $(1, 1) \in \mathcal{S}$ corresponds to the hyperbolic point in $\mathbb{D}(\hat{P})$, and hence any neighborhood of the hyperbolic point of $\mathbb{D}(\hat{P})$ is not a manifold.

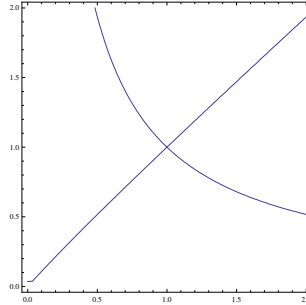


FIGURE 6. $8x - (5 + \sqrt{5})y - (6 - 2\sqrt{5})xy - (5 + \sqrt{5})x^2y + 8xy^2 = 0$

5.3. The condition (C2) is necessary. Let d be a fixed integer > 3 . We consider the compact hyperbolic Coxeter 3-polytope P shown in Figure 7. Here, if an edge is labeled d then its dihedral angle is $\frac{\pi}{d}$. Otherwise, its dihedral angle is $\frac{\pi}{2}$.

Obviously, $e_+ - 3 = 0$. However \hat{P} is not weakly orderable, since every facet in \hat{P} contains four edges of order 2.

Observe that P has the reflectional symmetry interchanging F and F' , ie the Coxeter 3-orbifold \hat{P} arising from P has order two symmetry about an embedded totally geodesic 2-dimensional suborbifold S . Consequently there are non-trivial

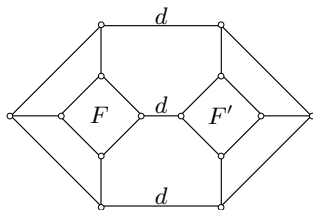


FIGURE 7. A compact hyperbolic Coxeter 3-polytope

deformations in $\mathbb{D}(\hat{P})$ obtained by *projective bendings* along S , and hence a neighborhood of the hyperbolic point in $\mathbb{D}(\hat{P})$ is *not* a manifold of dimension 0. See Choi, Hodgson and Lee [14] and Johnson and Millson [24] for the details.

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